Physical Principles in Biology
Biology 3550
Fall 2016

Lecture 10

Random Walks in Two Dimensions

Wednesday, 14 September

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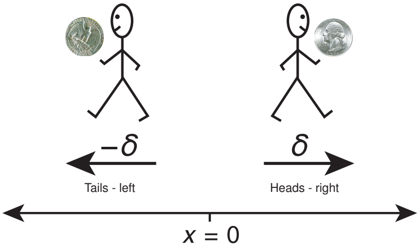
A Random Walk in One Dimension



1. Start at position $x = 0$.
2. Flip coin.
   - Heads, take step of length δ to the right.
   - Tails, take step of length δ to the left.
3. Repeat 2 another $(n-1)$ times.
4. Final position is $x(n)$.

Symbols Used in the Treatment of the 1-dimensional Random Walk

For a single random walk

- n: number of steps in the random walk.
- $x(i)$: position along the x-axis at the end of step i $(1 \leq i \leq n)$
- δ: length of each step (a positive value)
- δ_i: change in x for step i, either $+\delta$ or $-\delta$
- $p(+\delta)$: probability of a step to the right (towards larger x)
- $p(-\delta)$: probability of a step to the left, equal to $1 - p(+\delta)$
- $\langle\delta_i\rangle$: mean value of δ_i, over many steps (exp. value $E(\delta_i)$).
- $\langle\delta_i^2\rangle$: mean-square value of δ_i, over many steps (exp. value $E(\delta_i^2)$).

Symbols Used in the Treatment of the 1-dimensional Random Walk

For a large number of random walks

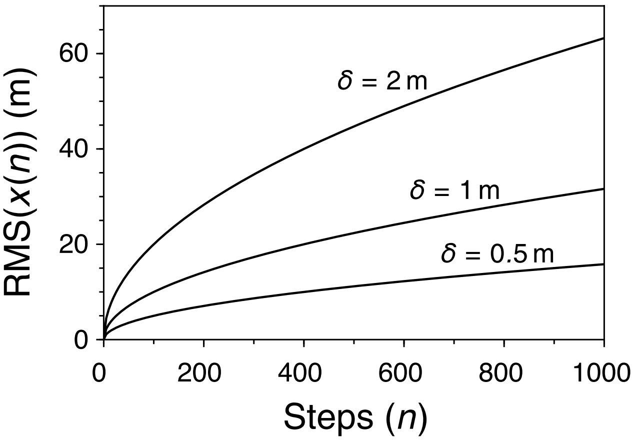
- N: total number of random walks
- $x(i)_j$: x-position of random walk j $(1 \leq j \leq N)$, after i $(1 \leq i \leq n)$ steps
- $\langle x(i) \rangle$: mean x-value after i steps, equal to $\left(\sum_{j=1}^{N} x(i)_j \right) / N$
- $\langle x(i)^2 \rangle$: mean-square x-value after i steps, equal to $\left(\sum_{j=1}^{N} x(i)_j^2 \right) / N$
- $\mathrm{RMS}(x(i))$: root-mean-square x-value after i steps, equal to $\sqrt{\langle x(i)^2 \rangle}$

Major Results for the 1-dimensional Random Walk

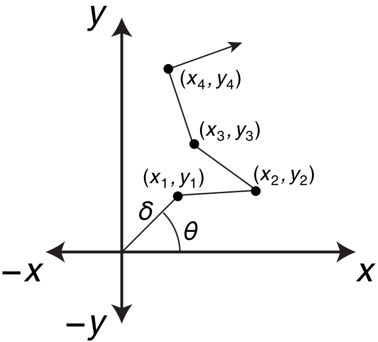
- Without constraints on $E(\delta_i)$
  - $\langle x(n) \rangle = nE(\delta_i)$
- If $E(\delta_i) = 0$ (unbiased choice of direction)
  - $\langle x(n) \rangle = 0$
  - $\langle x(n)^2 \rangle = nE(\delta_i^2) = n\delta^2$
  - $\mathrm{RMS}(x(n)) = \sqrt{\langle x(n)^2 \rangle} = \sqrt{n}\delta$

The Root-mean-square Displacement for a One-dimensional Random Walk

$$\mathrm{RMS}(x(n)) = \sqrt{\langle x(n)^2 \rangle} = \sqrt{n}\delta$$

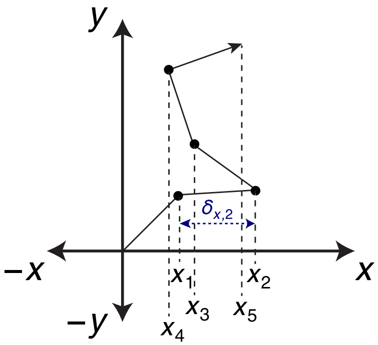


A Random Walk in Two Dimensions



1. Start at (x, y) coordinates (0,0).
2. Choose a random direction, defined by the angle θ from the x-axis.
3. Move distance δ in the chosen direction.
4. Repeat 2 and 3 another $(n-1)$ times.

A Random Walk in Two Dimensions

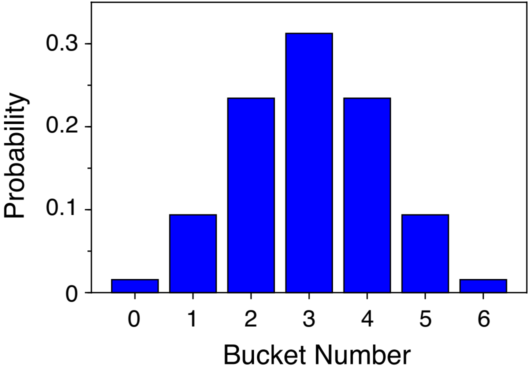


- x-coordinates represent a random walk along the x-axis.
- Can also describe a random walk along the y-axis (or any other direction).
- What are $\langle x_n \rangle$, $\langle x_n^2 \rangle$ and $\mathrm{RMS}(x_n)$?
- The change in x with each step, δ_x is not discrete!

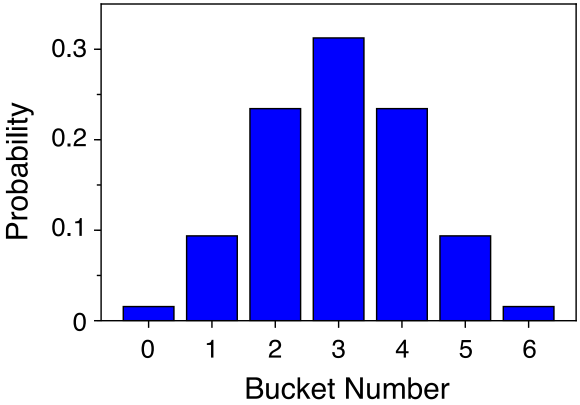
Discrete Probability Distribution Functions

- So far, we have dealt with events with a finite number of discrete outcomes and random variables with discrete values.
- The probability distribution functions can be viewed as tables or bar graphs

Bucket No.	Probability
0	1/64
1	6/64
2	15/64
3	20/64
4	15/64
5	6/64
6	1/64



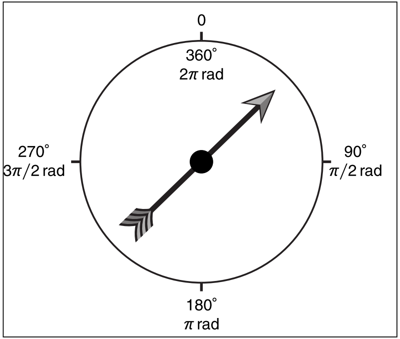
Graph For Discrete Probability Distribution Function



- Sum of probabilities = 1
- If the total width of bars is defined as 1, the total area of the bars is 1.

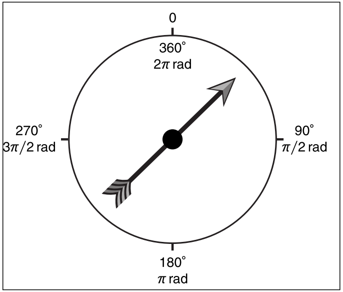
Introducing Continuous Probability Distribution Functions

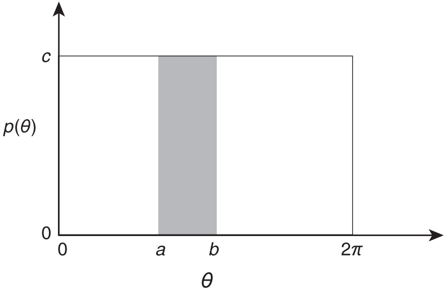
- A spinner to choose directions for the 2-dimensional random walk



- We could divide up the circle into a finite number of sectors.
    - Two sectors: Like flipping a coin
    - Six sectors: Like throwing a die
    - Lots of other possibilities
- OR, we can treat the result as a continuous variable from 0 to 2π rad

A Continuous Probability Distribution Function for the Spinner



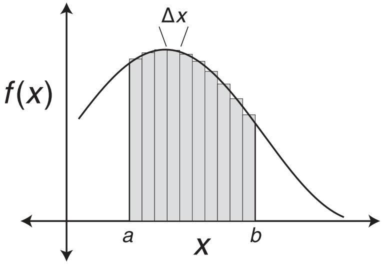


- θ is a continuous variable, with values from 0 to 2π.
- $p(\theta)$ is a function of θ, with a constant value, c, for all values of θ.
- Interpretation of $p(\theta)$: The integral

$$\int_a^b p(\theta) d\theta$$

is the probability that the spinner lands between the values a and b.

A Quick Refresher ons Integrals (as “area under the curve”)



- To approximate the area between the x-axis and the function $f(x)$, between $x = a$ and $x = b$:
  - Divide up the range $a \leq x \leq b$ into n segments $\Delta x = (b - a)/n$ wide.
  - Draw n rectangles Δx wide and $f(x_i)$ high.
  - Sum the areas of the rectangles

$$\text{area} = \sum_{i=1}^{n} f(x_i)\Delta x$$

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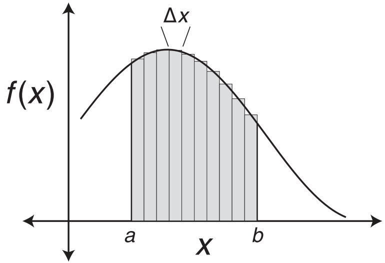

- Improve approximation by making Δx smaller (and n larger).
- If the function is "well behaved", Δx can be made infinitesimally small.
- The definite integral, from a to b with respect to x, is defined as:

$$\int_a^b f(x)dx = \lim_{\Delta x \to 0} \sum_{i=1}^{n} f(x_i)\Delta x$$

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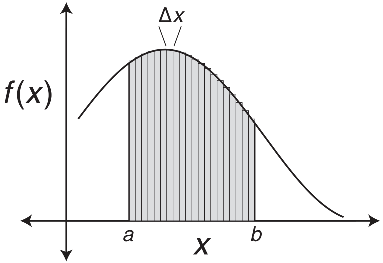

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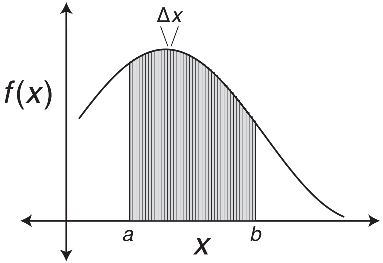

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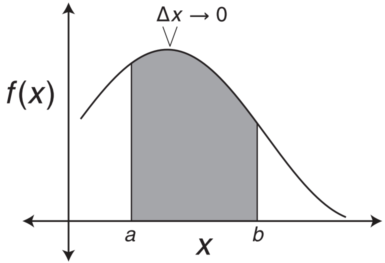

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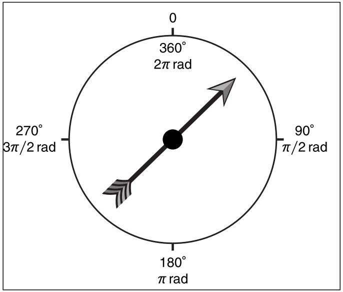
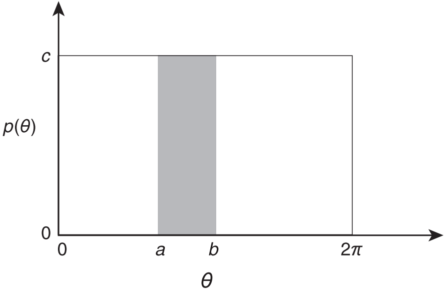
Back to the Continuous Probability Distribution Function (PDF) for the Spinner

- $p(\theta)$ is a function of θ, with a constant value, c, for all values of θ.



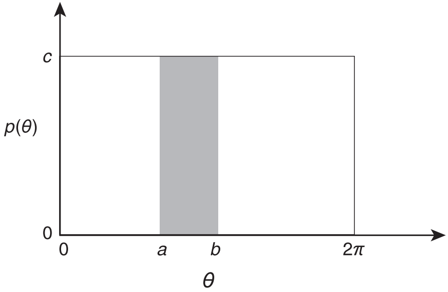


- The integral

$$\int_a^b p(\theta)d\theta$$

is the probability that the spinner lands between the values a and b.

An Important Constraint on a Continuous PDF



- To be properly "normalized", the integral

$$\int_0^{2\pi} p(\theta) d\theta$$

must equal 1.

- Equivalent to the requirement for a discrete PDF that the sum of all probabilities be equal to 1.

Normalizing the Spinner PDF

- $p(\theta) = c$ (a constant)

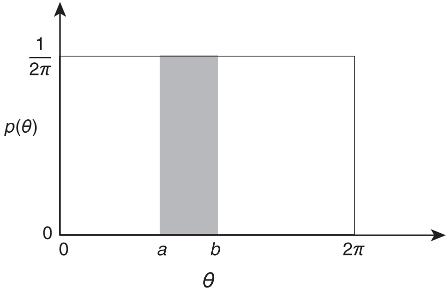
$$\int_0^{2\pi} p(\theta) d\theta = \int_0^{2\pi} c d\theta = 1$$

$$\int_0^{2\pi} c d\theta = c\theta \Big|_0^{2\pi} = (2\pi - 0)c = 1$$

- Solving for c

$$c = \frac{1}{2\pi}$$

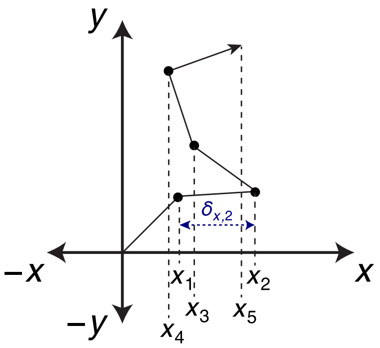
The Normalized Spinner PDF



- The function: $p(\theta) = 1/(2\pi)$
- The probability that θ lies between a and b:

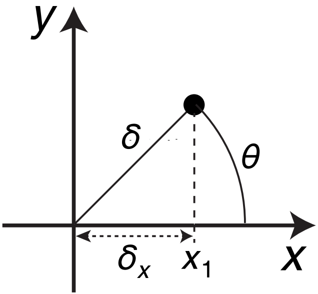
$$\int_a^b p(\theta) d\theta = \int_a^b \frac{1}{2\pi} d\theta = \frac{\theta}{2\pi}\Bigg|_a^b = \frac{b-a}{2\pi}$$

Back to the 2-dimensional Random Walk



- The first step is to describe the 1-dimensional walk along the x-axis.
- What are $\langle x_n \rangle$, $\langle x_n^2 \rangle$ and $\mathrm{RMS}(x_n)$?
- We need to know the expected value for each change in x, δx.
- The change in x with each step, δ_x is not discrete.

A Continuous PDF for δ_x

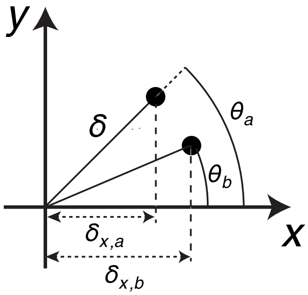


- δ_x is a continuous random variable.
- δ_x is related to θ according to:

$$\cos(\theta) = \frac{\delta_x}{\delta}$$

$$\delta_x = \delta \cos(\theta)$$

A Change of Coordinates



- The probability that δ_x lies between $\delta_{x,a}$ and $\delta_{x,b}$ is the same as the probability that θ lies between θ_b and θ_a.

$$\int_{\delta_{x,a}}^{\delta_{x,b}} p(\delta_x)\, d\delta_x = \int_{\theta_b}^{\theta_a} p(\theta)\, d\theta = \int_{\theta_b}^{\theta_a} \frac{1}{2\pi} d\theta$$

The Expected Value of δ_x

- For a discrete random variable, x, with discrete PDF, $p(x)$, the expected value is:

$$E(x) = \sum_{i=1}^{n} xp(x)$$

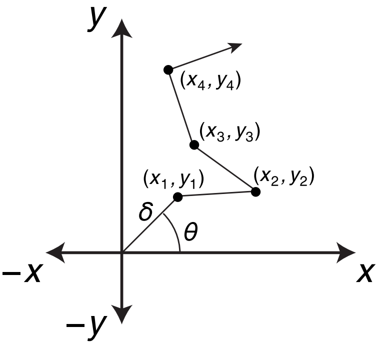
- For a continuous random variable, x, with range $x_1 \leq x \leq x_2$ and continuous PDF, $p(x)$, the expected value is:

$$E(x) = \int_{x_1}^{x_2} xp(x)dx$$

- For δ_x:

$$E(\delta_x) = \int_0^{2\pi} \delta \cos(\theta)p(\theta)d\theta = \int_0^{2\pi} \delta \cos(\theta)\frac{1}{2\pi}d\theta = \left.\frac{\delta \sin(\theta)}{2\pi}\right|_{\theta=0}^{\theta=2\pi} = 0$$

The Expected Value of δ_x



- For each step, δ_x is as likely to have the value $-a$ as the value $+a$.
- It makes sense that $E(\delta_x) = 0$.