Physical Principles in Biology
Biology 3550
Fall 2016

Lecture 11

Random Walks in Two Dimensions - Part 2

Friday, 16 September

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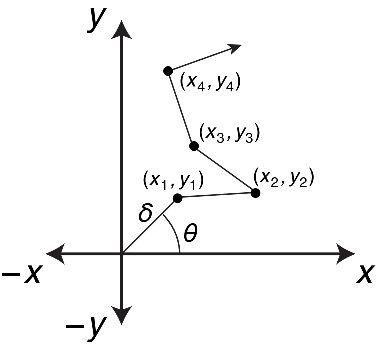
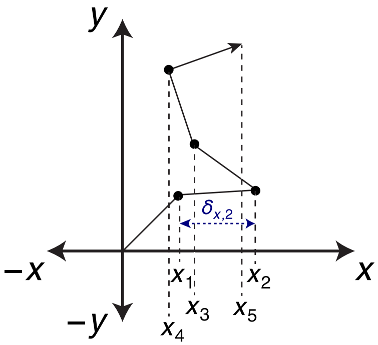
A Random Walk in Two Dimensions



1. Start at (x, y) coordinates (0,0).
2. Choose a random direction, defined by the angle θ from the x-axis.
3. Move distance δ in the chosen direction.
4. Repeat 2 and 3 another $(n-1)$ times.

A Random Walk in Two Dimensions



- x-coordinates represent a random walk along the x-axis.
- Can also describe a random walk along the y-axis (or any other direction).
- What are $\langle x_n \rangle$, $\langle x_n^2 \rangle$ and $\mathrm{RMS}(x_n)$?
- Need the expected values of δ_x and δ_x^2, $E(\delta_x)$ and $E(\delta_x^2)$

Clicker Question #1

What is the expected value of δ_x, $E(\delta_x)$?

1. 0
2. $1/2$
3. $\delta^2/2$
4. $\delta\cos(\theta)$
5. 2π

The Expected Value of δ_x

- For a discrete random variable, x, with discrete PDF, $p(x)$, the expected value is:

$$E(x) = \sum_{i=1}^{n} xp(x)$$

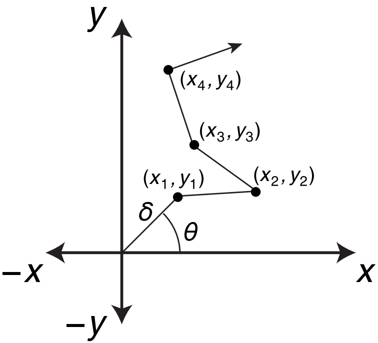
- For a continuous random variable, x, with range $x_1 \leq x \leq x_2$ and continuous PDF, $p(x)$, the expected value is:

$$E(x) = \int_{x_1}^{x_2} xp(x)dx$$

- For δ_x:

$$E(\delta_x) = \int_0^{2\pi} \overbrace{\delta\cos(\theta)}^{\delta_x} p(\theta)d\theta = \int_0^{2\pi} \delta\cos(\theta)\frac{1}{2\pi}d\theta = \frac{\delta\sin(\theta)}{2\pi}\Bigg|_{\theta=0}^{\theta=2\pi} = 0$$

The Expected Value of δ_x



- For each step, δ_x, is as likely to have the value $-a$ as the value $+a$.
- It makes sense that $E(\delta_x) = 0$.

The Expected Value of δ_x^2

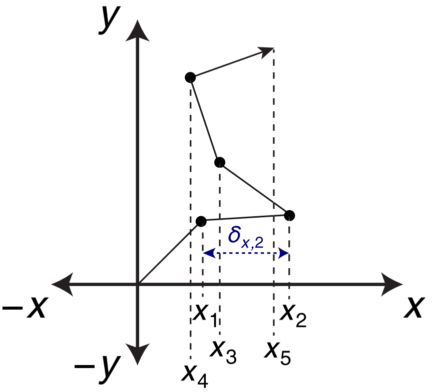
$$E(\delta_x^2) = \int_0^{2\pi} (\delta \cos(\theta))^2 p(\theta) d\theta$$

$$= \int_0^{2\pi} (\delta \cos(\theta))^2 \frac{1}{2\pi} d\theta$$

$$= \left. \frac{\delta^2 \left(\frac{\sin(2\theta)}{2} + \theta \right)}{4\pi} \right|_{\theta=0}^{\theta=2\pi} = \frac{\delta^2}{2}$$

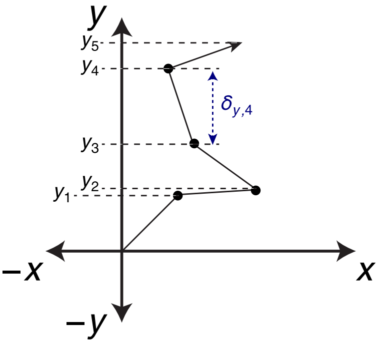
- Now we can calculate $\langle x_n \rangle$, $\langle x_n^2 \rangle$ and $\mathrm{RMS}(x_n)$ for the 2-dimensional random walk!

The Random Walk Along the x-axis
(during the 2-dimensional random walk)



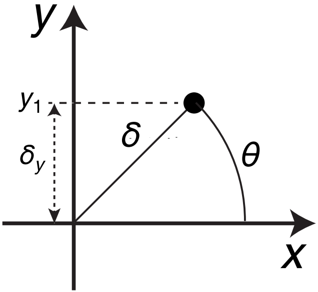
- $E(\delta_x) = 0$
- $E(\delta_x^2) = \delta^2/2$
- $\langle x_n \rangle = nE(\delta_x) = 0$
- $\langle x_n^2 \rangle = nE(\delta_x^2) = n\delta^2/2$
- $\text{RMS}(x_n) = \sqrt{\langle x_n^2 \rangle} = \sqrt{n/2}\delta$
- Allowing motion into a second dimension reduces the average (RMS) displacement (along the x-axis) by $\sqrt{2}$

A Random Walk in Two Dimensions



- y-coordinates represent a random walk along the y-axis.
- What are $\langle y_n \rangle$, $\langle y_n^2 \rangle$ and $\mathrm{RMS}(y_n)$?
- Need the expected values of δ_y and δ_y^2, $E(\delta_y)$ and $E(\delta_y^2)$

A Continuous PDF for δ_y

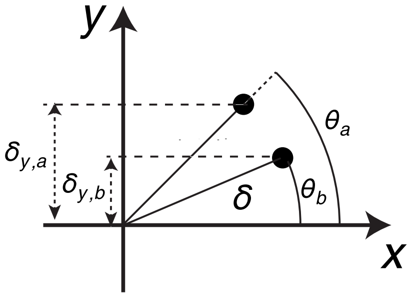


- δ_y is a continuous random variable.
- δ_y is related to θ according to:

$$\sin(\theta) = \frac{\delta_y}{\delta}$$

$$\delta_y = \delta \sin(\theta)$$

A Change of Coordinates



- The probability that δ_y lies between $\delta_{y,a}$ and $\delta_{y,b}$ is the same as the probability that θ lies between θ_a and θ_b.

$$\int_{\delta_{y,a}}^{\delta_{y,b}} p(\delta_y)\, d\delta_y = \int_{\theta_a}^{\theta_b} p(\theta)\, d\theta = \int_{\theta_a}^{\theta_b} \frac{1}{2\pi} d\theta$$

Clicker Question #2

What is the expected value of δ_y, $E(\delta_y)$?

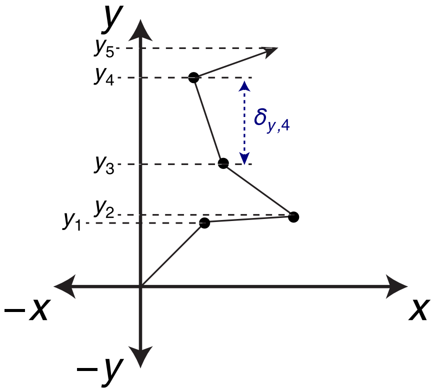
1. 0
2. $1/2$
3. $\delta^2/2$
4. $\delta \cos(\theta)$
5. 2π

Clicker Question #3

What is the expected value of δ_y^2, $E(\delta_y^2)$?

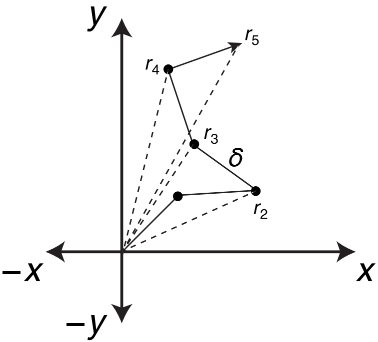
1. 0
2. $1/2$
3. $\delta^2/2$
4. $\delta\cos(\theta)$
5. 2π

The Random Walk Along the y-axis
(during the 2-dimensional random walk)



- $E(\delta_y) = 0$
- $E(\delta_y^2) = \delta^2/2$
- $\langle y_n \rangle = nE(\delta_y) = 0$
- $\langle y_n^2 \rangle = nE(\delta_y^2) = n\delta^2/2$
- $\mathrm{RMS}(y_n) = \sqrt{\langle y_n^2 \rangle} = \sqrt{n/2}\delta$
- **On average**, the walks along the y-axis are just like the walks along the x-axis.
- But each 2-dimensional walk will likely be different along the two axes.

Distance from the Starting Point



- r_i is the distance from the starting point to the position after step i.
- What are $\langle r_n \rangle$, $\langle r_n^2 \rangle$ and $\mathrm{RMS}(r_n)$?

The Expected Value for r^2

- For a single random walk:

$$r_n^2 = x_n^2 + y_n^2$$

- For two independent random variables, A and B

$$E(A + B) = E(A) + E(B)$$

- The expected value of r_n^2:

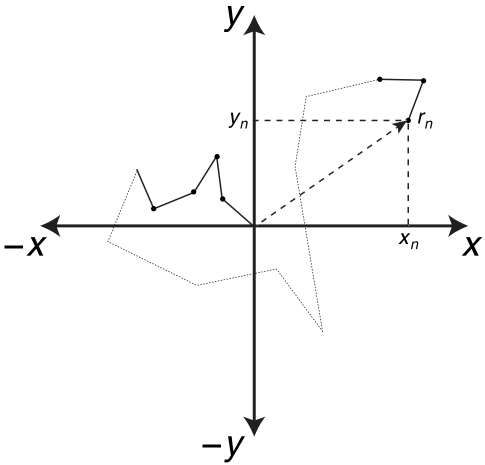
$$E(r_n^2) = E(x_n^2) + E(y_n^2)$$

- From before:

$$E(x_n^2) = n\delta^2/2 \quad \text{and} \quad E(y_n^2) = n\delta^2/2$$

$$E(r_n^2) = E(x_n^2) + E(y_n^2) = n\delta^2$$

$$\langle r_n^2 \rangle = n\delta^2 \quad \text{and} \quad \mathrm{RMS}(r_n) = \sqrt{n}\delta$$

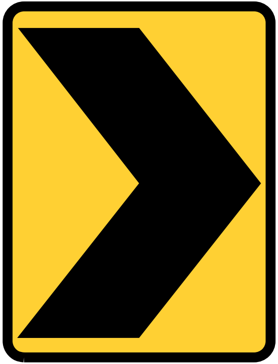


Major Results for a Two-Dimensional Random Walk

For n steps of length δ:

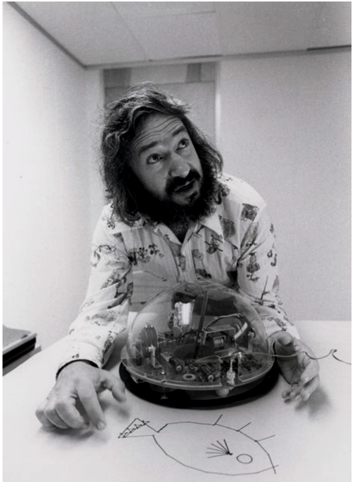
- Displacement along the x- and y-axes (or any other direction):
  - Mean displacement: $\langle x_n \rangle = \langle y_n \rangle = 0$.
  - Mean square displacement: $\langle x_n^2 \rangle = \langle y_n^2 \rangle = n\delta^2/2$
  - mean RMS displacement: $\mathrm{RMS}(x_n) = \sqrt{n/2}\,\delta$
- Distance from starting point, r:
  - Mean square displacement: $\langle r_n^2 \rangle = n\delta^2$
  - mean RMS displacement: $\mathrm{RMS}(r_n) = \sqrt{n}\,\delta$
  - Mean displacement: ?

Warning!



Direction Change

Seymour Papert 1928–2016

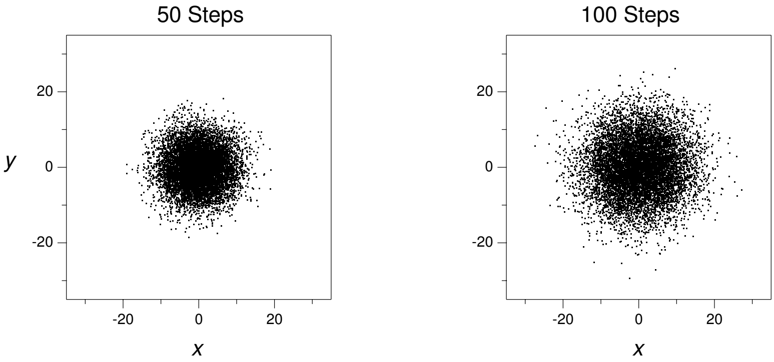


- Education researcher
- Computer scientist
- Co-developer of the Logo computer language

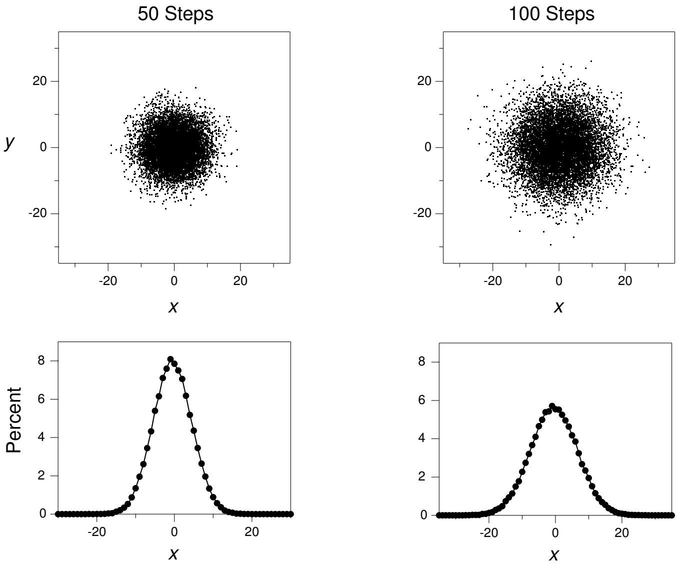
http://www.nytimes.com/2016/08/02/technology/seymour-papert-88-dies-saw-educations-future-in-computers.html?_r=0

http://el.media.mit.edu/logo-foundation/

End Points for 2-d Random Walks, Step length = 1



Final x-Coordinate for 2-d Random Walks



Final Distance from Origin for 2-d Random Walks