Physical Principles in Biology
Biology 3550
Fall 2016

Lecture 12

Random Walks in Two Dimensions - Part 3

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Major Results for a Two-Dimensional Random Walk

For n steps of length δ:

- Displacement along the x- and y-axes (or any other direction):
  - Mean displacement: $\langle x_n \rangle = \langle y_n \rangle = 0$.
  - Mean square displacement: $\langle x_n^2 \rangle = \langle y_n^2 \rangle = n\delta^2/2$
  - mean RMS displacement: $\mathrm{RMS}(x_n) = \sqrt{n/2}\delta$
- Distance from starting point, r:
  - Mean square displacement: $\langle r_n^2 \rangle = n\delta^2$
  - mean RMS displacement: $\mathrm{RMS}(r_n) = \sqrt{n}\delta$
  - Mean displacement: ?

An Implication

- Number of steps = n.
- Length of steps = δ
- Total distance: $D_t = n\delta$.
- If total distance is fixed and δ is changed:

$$n = D_t/\delta$$

$$\begin{aligned}\mathrm{RMS}(r) &= \sqrt{n}\delta \\ &= \sqrt{D_t/\delta}\cdot\delta \\ &= \sqrt{D_t}\sqrt{\delta}\end{aligned}$$

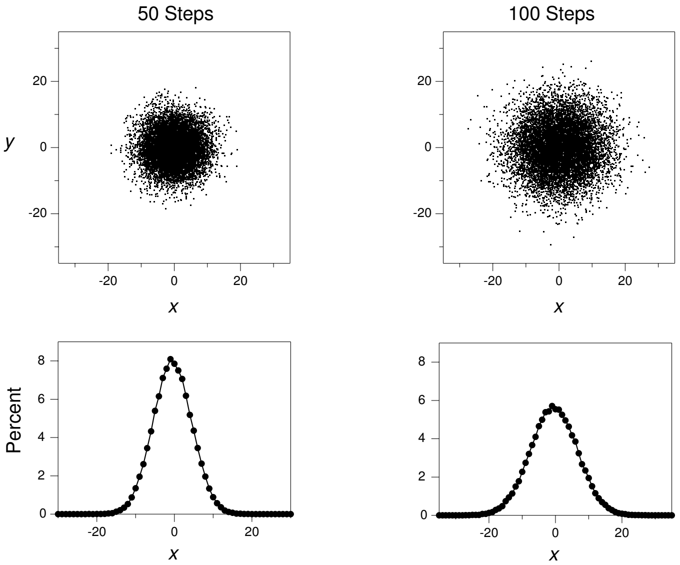
Average distance from start to end increases with step length.

Clicker Question #1

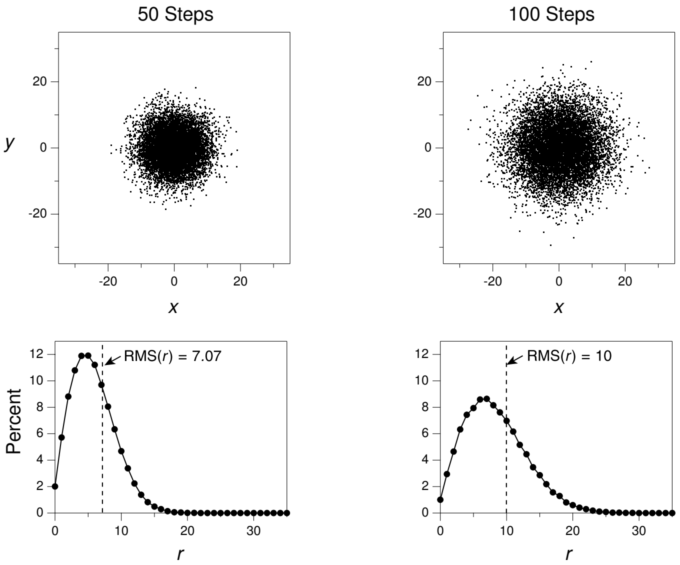
For a 2-dimensional random walk of a total distance of 100 m and a step length of 5 m, which (if any) of the following are correct?

1. $\mathrm{RMS}(r) \approx 50\,\mathrm{m}$
2. $\langle r^2 \rangle \approx 500\,\mathrm{m}^2$
3. $\langle r^2 \rangle \approx 250\,\mathrm{m}^2$
4. $\mathrm{RMS}(r) \approx 22\,\mathrm{m}$
5. None of the above

Final x-Coordinate for 2-d Random Walks



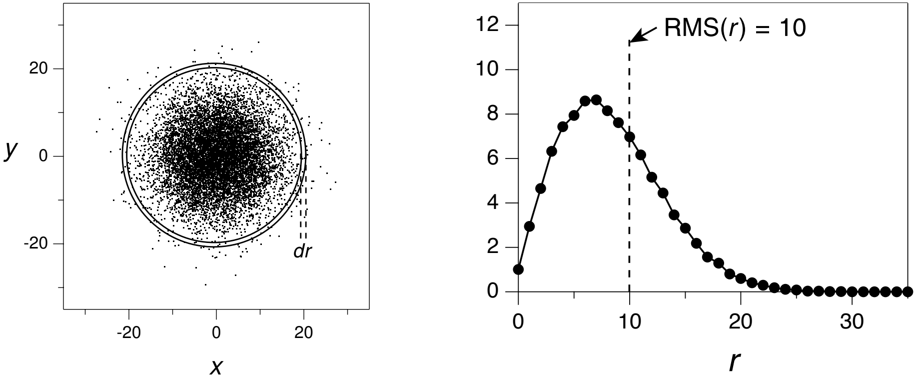
Final Distance from Origin for 2-d Random Walks



- Why isn't the peak at $r = 0$?

Why Isn't the Peak at $r = 0$

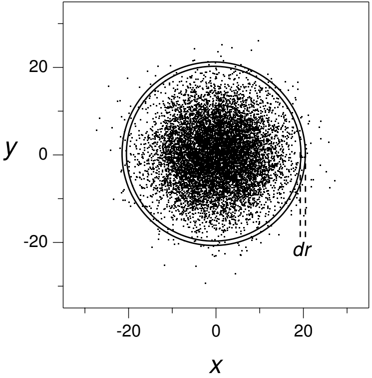
- $\int_{r_a}^{r_b} p(r)dr =$ probability that the walk endpoint lies between r_a and r_b.



- $p(r)dr =$ probability that the endpoint lies in the annulus (ring) dr thick.

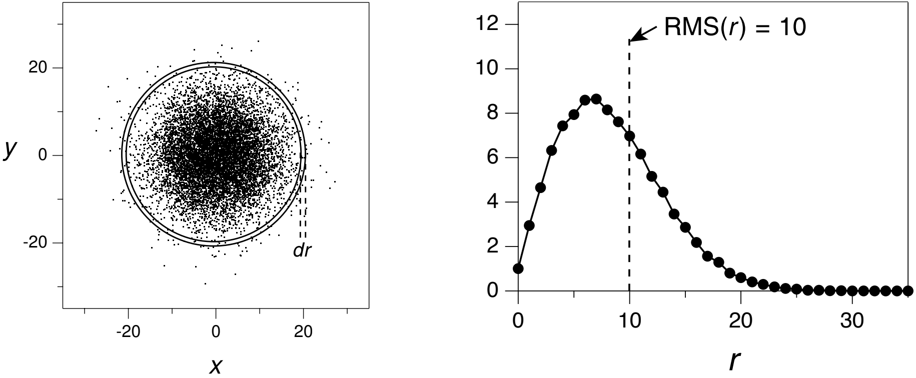
Clicker Question #2

What is the area of the annulus?



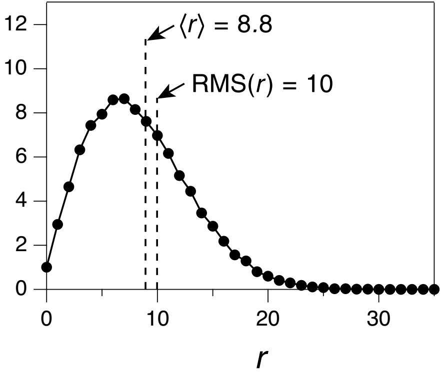
1. πr^2
2. πdr^2
3. $2\pi r$
4. $2\pi dr$
5. $2\pi r dr$

Why isn't the Peak at $r = 0$



- The probability, $p(r)dr$, is proportional to the area of the annulus.
- The area increases with r: $A = 2\pi r dr$.
- The density of endpoints decreases with r.
- The two effects balance at the peak of the distribution.

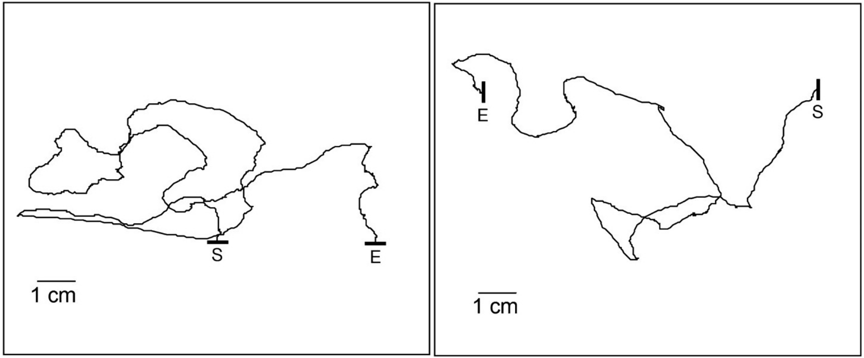
Why isn't the Peak at $r = \langle r \rangle$ or $r = \mathrm{RMS}(r)$?



$$\langle r \rangle = E(r) = \int_0^{r_{\max}} r p(r) dr$$

$$\mathrm{RMS}(r) = \sqrt{E(r^2)} = \sqrt{\int_0^{r_{\max}} r^2 p(r) dr}$$

Ants on a Walk for Food



Brachymyrmex depilis
(25 s)

Dorymyrmex insanus
(21 s)

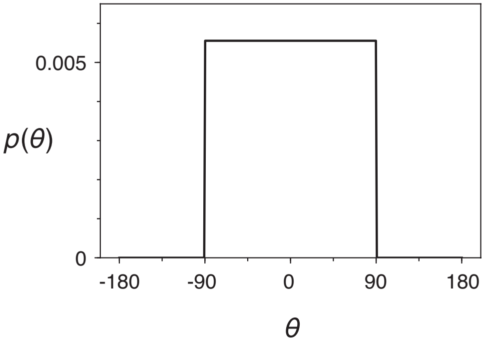
- Do either look like a random walk?

Pearce-Duvet, J. M. C., Elemens, C. P. H. & Feener, D. H. (2011). Walking the line: search behavior and foraging success in ant species. *Behavioral Ecology*, 22, 501–509.
`http://dx.doi.org/10.1093/beheco/arr001`

Restricted Turn Angles Described by Continuous PDFs

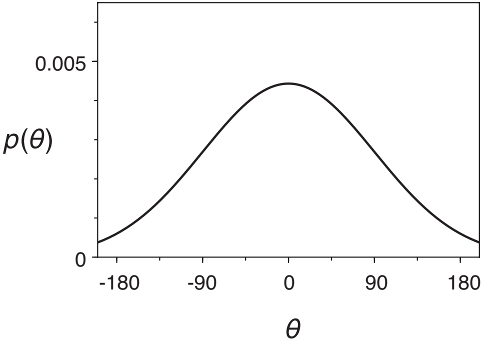
Turn angle, θ, defined relative to current direction

A top-hat distribution



$$p(\theta) = \begin{cases} 1/180 & \text{if } |\theta| \leq 90 \\ 0 & \text{otherwise} \end{cases}$$

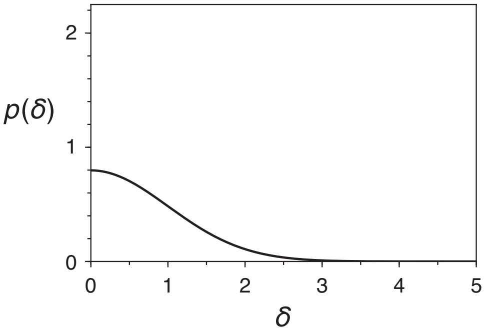
A Gaussian distribution



$$p(\delta) = \frac{1}{90\sqrt{2\pi}} e^{-\delta^2/(2\cdot 90^2)}$$

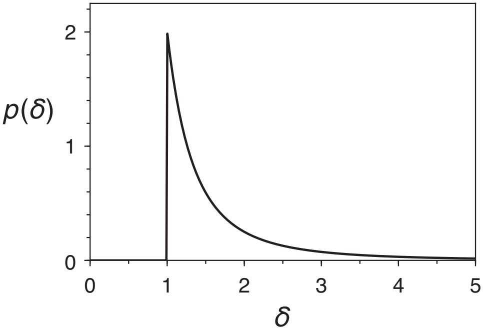
Variable Step Lengths Described by Continuous PDFs

A half-Gaussian distribution



$$p(\delta) = \begin{cases} \frac{1}{\sqrt{2\pi}} e^{-\delta^2/2} & \text{if } \delta \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

A “long-tailed” distribution



$$p(\delta) = \begin{cases} \frac{2}{\delta^3} & \text{if } \delta \geq 1 \\ 0 & \text{otherwise.} \end{cases}$$

(A Pareto Distribution)