Physical Principles in Biology
Biology 3550
Fall 2016

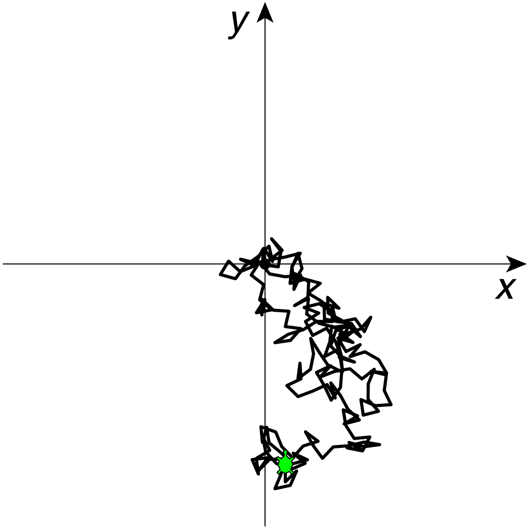
Lecture 13

A Bit More on Random Walks and the Gaussian (Normal) Probability Distribution

Wednesday, 21 September

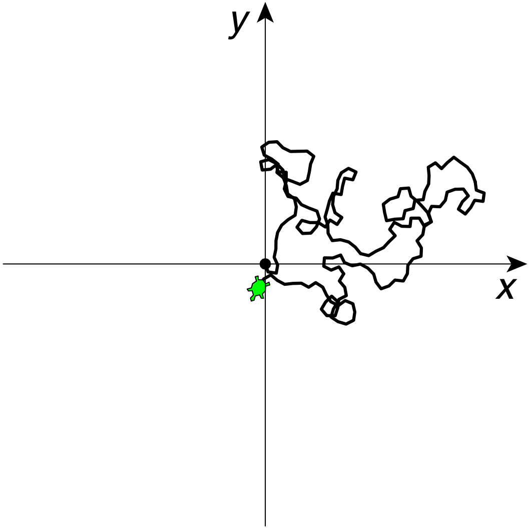
©David P. Goldenberg
University of Utah
goldenberg@biology.utah.edu

A ‘Plain’ Random Walk



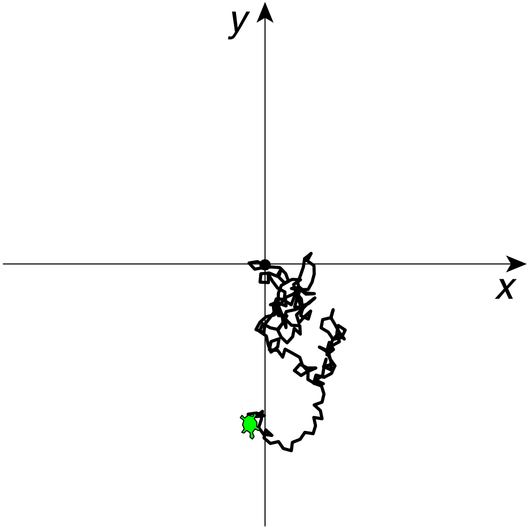
- Step length = 20
- No. steps = 200

A “Correlated” Random Walk



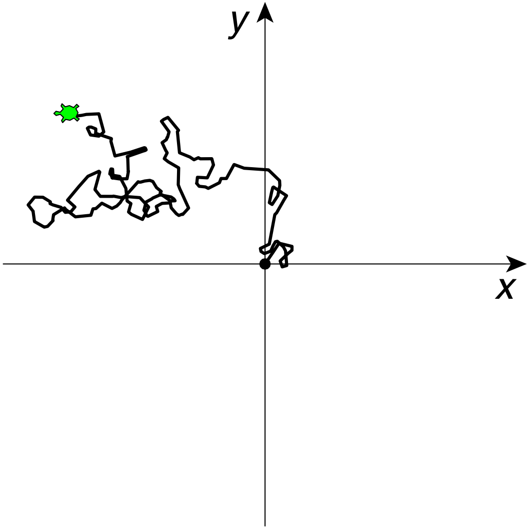
- Turn angle restricted to -90° to 90°
- Step length = 8
- No. steps = 200

A “Correlated” Random Walk



- Gaussian distribution of turn angles
- Step length = 8
- No. steps = 200

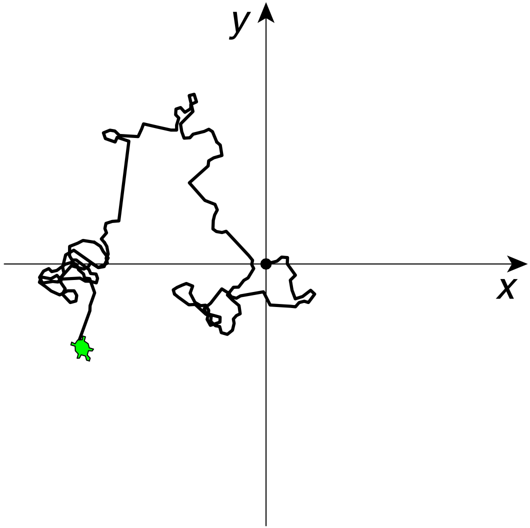
A Random Walk With a Distribution of Step Lengths



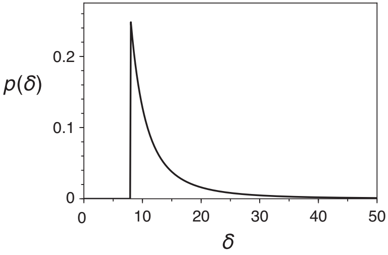
- Turn angle restricted to -90° to 90°
- Half gaussian distribution of step lengths
- No. steps = 200

A “Lévy Flight”

A random walk with a “heavy-tailed” distribution of step lengths



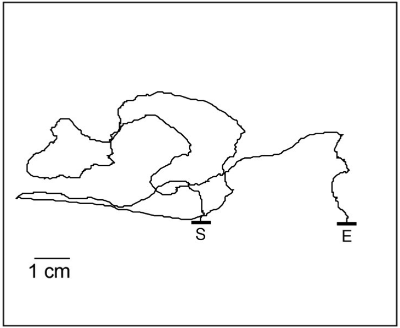
- Turn angle restricted to -90° to 90°
- Pareto distribution of step lengths



- No. steps = 200

Clicker Question #1

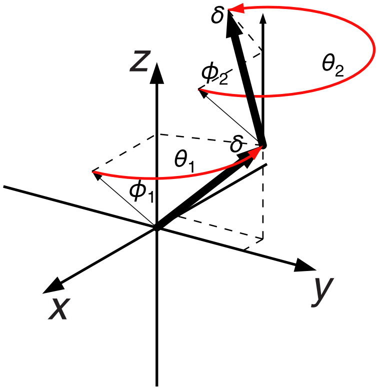
What does the ant walk most resemble?



Brachymyrmex depilis
(25 s)

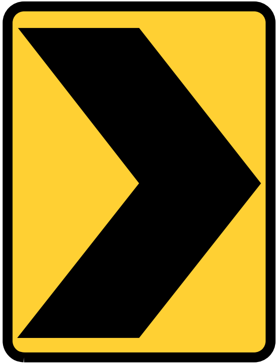
1. A plain random walk
2. A correlated random walk
3. A Lévy flight

Description of a Three-dimensional Random Walk



- Each step is defined by a tilt from the local z-axis (ϕ_i) and a rotation around the z-axis (θ_i).
- The end of each step lies on a sphere of radius δ.
- $\langle r^2 \rangle = n\delta^2$, and $\mathrm{RMS}(r) = \sqrt{n}\delta$, just like in one and two dimensions.

Warning!



Direction Change

From the Binomial to the Gaussian Distribution

Plinko buckets (n even)

$k = 0 \quad 1 \quad \cdots\cdots \quad n/2 \quad \cdots\cdots \quad n-1 \quad n$

$x = k - \frac{n}{2} = -\frac{n}{2} \quad -\frac{n}{2}+1 \quad 0 \quad \frac{n}{2}-1 \quad \frac{n}{2}$

$$p(k) = \binom{n}{k}\left(\frac{1}{2}\right)^n = \frac{n!}{(n-k)!(k)!}\frac{1}{2^n}$$

$$p(x) = \binom{n}{x+n/2}\left(\frac{1}{2}\right)^n = \frac{n!}{(n/2-x)!(n/2+x)!}\frac{1}{2^n}$$

for n odd or even

From the Binomial to the Gaussian Distribution

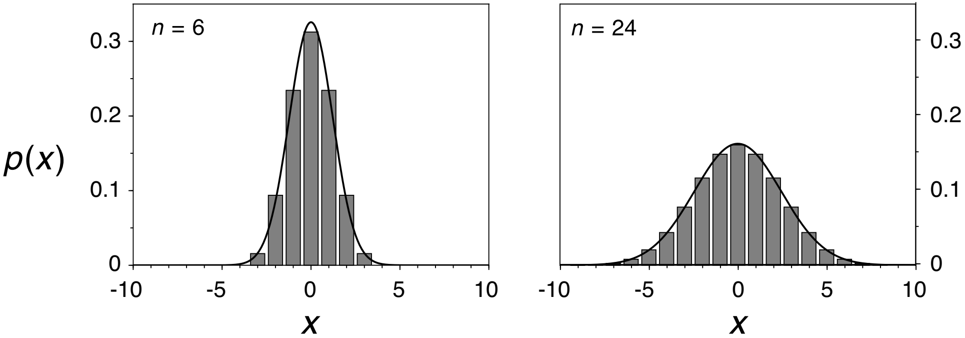
- As n gets large (Sterling's approximation):

$$n! \longrightarrow \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

- and:

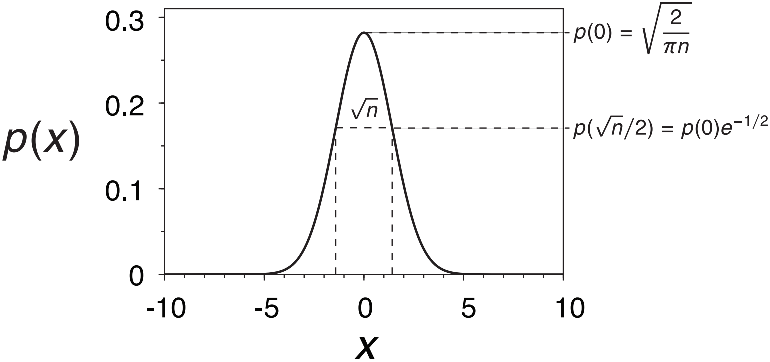
$$p(x) = \frac{n!}{(n/2 - x)!(n/2 + x)!}\frac{1}{2^n} \longrightarrow \sqrt{\frac{2}{\pi n}}e^{-2x^2/n}$$

From the Binomial to the Gaussian Distribution



- Bars represent discrete binomial distribution.
- Curves represent continuous Gaussian (normal) distribution.

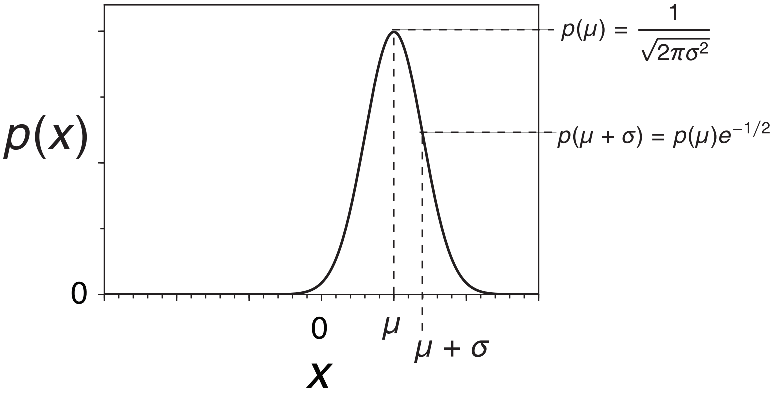
Gaussian Distribution for $n = 8$



$$p(x) = \sqrt{\frac{2}{\pi n}} e^{-2x^2/n}$$

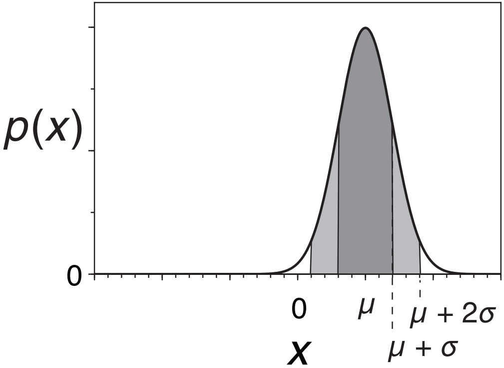
The More General Form of the Gaussian PDF

- $p(x) = \sqrt{\frac{1}{2\pi\sigma^2}} e^{-(x-\mu)^2/(2\sigma^2)}$



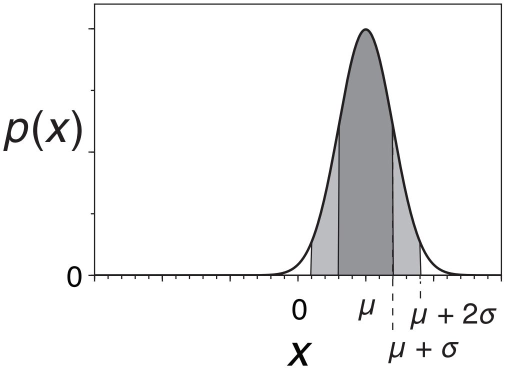
- μ is the expected value of x, $\langle x \rangle$, and represents the displacement of the distribution from 0.

The Meaning of σ



- σ^2 is called the "variance" and is the mean-square deviation from the mean of values drawn from the distribution: $\sigma^2 = \langle (x - \mu)^2 \rangle$
- σ is called the "standard deviation" and is the root-mean-square deviation from the mean: $\sigma = \sqrt{\langle (x - \mu)^2 \rangle} = \mathrm{RMS}(x - \mu)$

The Meaning of σ



- For a normal distribution, $\approx 68\%$ of values drawn from the distribution are expected to lie between $\mu - \sigma$ and $\mu + \sigma$.

$$\int_{\mu-\sigma}^{\mu+\sigma} p(x)dx \approx 0.6827$$

- $\approx 95\%$ of values are expected to lie between $\mu - 2\sigma$ and $\mu + 2\sigma$.