Physical Principles in Biology
Biology 3550
Fall 2016

Lecture 15

A Bit More on Statistics and Error Analysis

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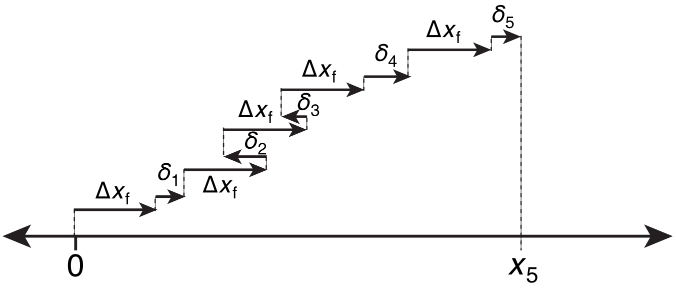
A New Kind of Random Walk

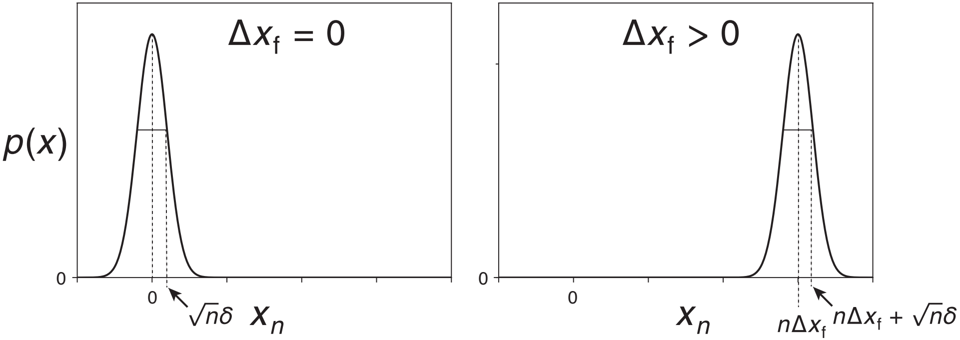


- Each step, i, has two parts:
  1. A fixed displacement of Δx_f, always of the same magnitude and sign.
  2. A random displacement of δ_i, such that:

$$\langle \delta_i \rangle = 0$$
$$\langle \delta_i^2 \rangle = \delta^2$$

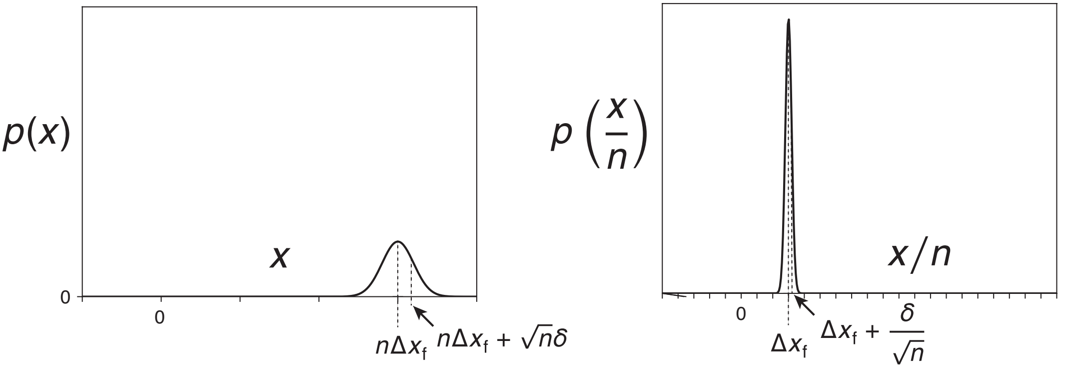
- Each random walk consists of n two-step steps.

Distribution of Endpoints, x_n, for N Walks



- $\langle x_n \rangle = n\Delta x_f$
- $\langle (x_n - n\Delta x_f)^2 \rangle = n\delta^2$
- A normal distribution, with $\mu = n\Delta x_f$ and $\sigma = \sqrt{n}\delta$.
- As n increases, the distribution moves to the left *and* spreads out.

Suppose that we divide x by n



- $\langle x_n/n \rangle = \Delta x_f$
- $\langle (x_n/n - \Delta x_f)^2 \rangle = \delta^2/n$
- A normal distribution, with $\mu = n\Delta x_f$ and $\sigma = \delta/\sqrt{n}$.
- The longer the walk, the narrower the distribution of x/n.

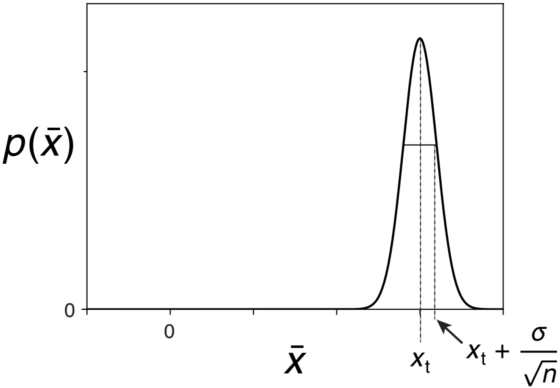
Something Similar to Our Funny Random Walk

- Make n repeated measurements of some physical quantity, x, such as the mass of an object.
- Each measurement, x_i, is the sum of:
  - The "true" value of the measurement, x_t.
  - A random error, δ_i, with $\langle\delta\rangle = 0$ and a standard deviation, $\sigma = \mathrm{RMS}(\delta)$.
- Sum all of the measured values:

$$\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} (x_\mathrm{t} + \delta_i)$$

- Divide the sum by the number of measurements, n.
  Call this $\bar{x}$.
- How would we interpret the result: $\bar{x} = \left(\sum x_i\right)/n$?
- What would we expect to find if we were to repeat the series of n measurements a large number of times?

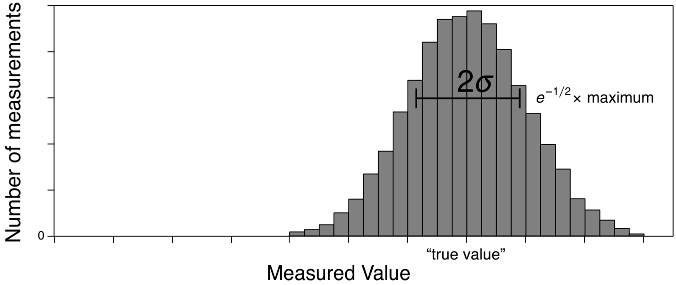
Distribution of Averages of Experimental Measurements



- The most probable value of $\bar{x}$ is x_t, the “true” value of x.
- The width of the distribution is proportional to σ, the standard deviation of the experimental errors.
- The width of the distribution decreases with $\sqrt{n}$.

Estimating the True Value, x_t, and the Std. Dev. of Errors, σ, from n Measurements of x

- The best* estimate of the true value is the mean of the experimental measurements, $\langle x \rangle$, (also written $\bar{x}$).
- Could estimate σ from a histogram of measured values:



This would take a lot of measurements!
(The simulation used 10,000.)

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*"Best" means most likely to give the correct answer.

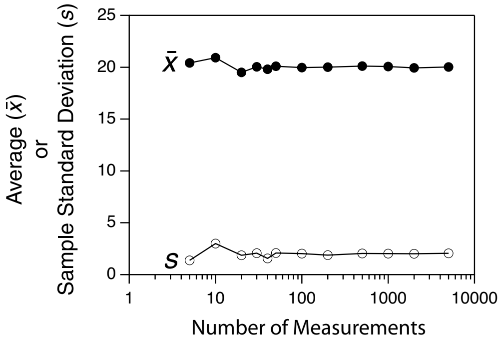
A Formula for Estimating σ from n measurements

- The **sample** standard deviation:

$$s = \sqrt{\frac{\sum(x_i - \bar{x})^2}{n-1}}$$

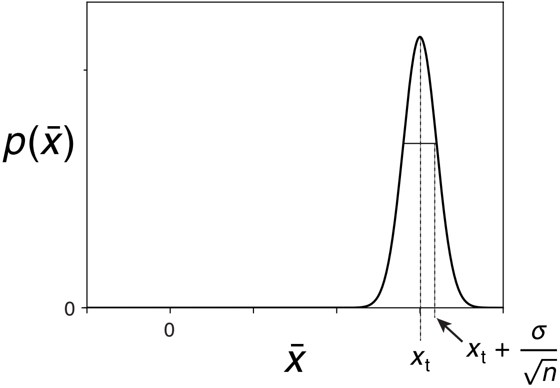
- s is an estimate of σ, the “true” standard deviation of the errors, and is almost the same as $\mathrm{RMS}(x - \bar{x})$.
- Why $n - 1$ instead of n in the denominator?
  - $\bar{x}$ is only an estimate of x_t.
  - $\bar{x}$ is the estimate of x_t that minimizes the sum $\sum(x_i - x_e)^2$.
  - If $\bar{x}$ is different from x_t, the expression for s with n in the denominator will under estimate σ.
  - Using $n - 1$ *almost* corrects for this.

Estimates Improve With More Measurements (A Simulation)



- Estimate of true value ($\bar{x}$) approaches a limiting value (20)
- Estimate of standard deviation (s) approaches a limiting value (2)
- s doesn't approach zero.
- The uncertainty in the estimate of $\bar{x}$ <u>does</u> decrease with n.

Distribution of Averages of Experimental Measurements

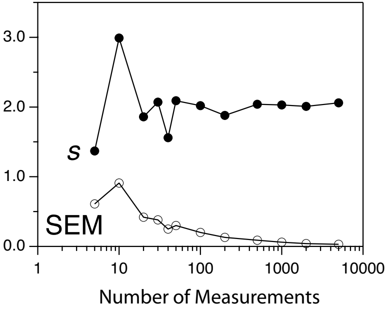


- $p(\bar{x})$ is the probability of observing a particular value of $\bar{x}$ for a given set of measurements.
- The probable deviation of $\bar{x}$ from the true value, x_t, is proportional to σ, the standard deviation of the experimental errors.
- The probable deviation of $\bar{x}$ from the true value, x_t, decreases with $\sqrt{n}$.

Another Useful Statistic: The Standard Error of the Mean (SEM)

$$\text{SEM} = \sqrt{\frac{\sum(x-\bar{x})^2}{(N-1)N}} = s/\sqrt{N}$$

- The standard error of the mean is an estimate of σ/n and represents the uncertainty in the estimate of the mean, $\bar{x}$
- The uncertainty in $\bar{x}$ decreases with more measurements.



Sample Standard Deviation vs. Standard Error of the Mean

- The **sample** standard deviation:

$$s = \sqrt{\frac{\sum(x_i - \bar{x})^2}{n - 1}}$$

  Provides an estimate of the uncertainty in the individual measurements.

- The standard error of the mean:

$$\text{SEM} = \sqrt{\frac{\sum(x - \bar{x})^2}{(N - 1)N}} = s/\sqrt{N}$$

  Provides estimate of the precision of $\bar{x}$ as an estimate of the true value.

Altman, D. G. & Bland, J. M. (2005). Standard deviations and standard errors. *BMJ*, 331, 903. `http://dx.doi.org/10.1136/bmj.331.7521.903`

Accuracy vs. Precision

- How "true" is the "true value"?
  - x_t is only the true value for the particular experimental set up, including measuring devices.
  - Practically, x_t is the limiting value of of $\bar{x}$ as n becomes very large.
  - There is no guarantee that x_t from one experimental setup is the same as from another setup.
- The sample standard deviation, s, is a measure of the precision (reproducibility) of individual measurements.
- The standard error of the mean, SEM, is a measure of the precision of the set of n measurements.
- Evaluating accuracy requires comparison to an external standard, like the international prototype kilogram stored in a French vault.