Physical Principles in Biology
Biology 3550
Fall 2016

Lecture 7

More on Plinko Probabilities

Wednesday, 7 September

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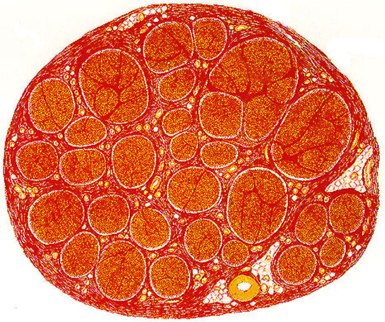
Human Sciatic Nerve



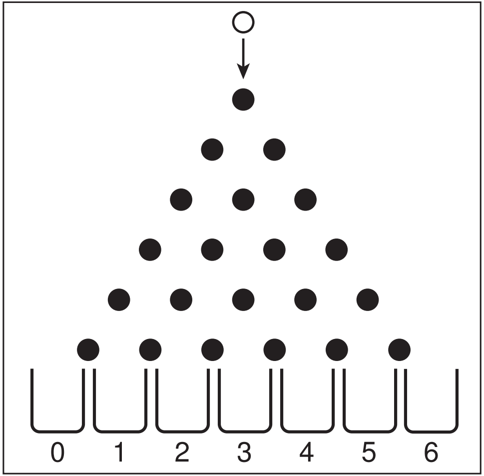
"Cross-section through a myelinated nerve (*e.g.* sciatic) showing individual nerve bundles each consisting of many fibres."

- Comparing linear dimensions: $\dfrac{10\,\text{mm}}{15\,\mu\text{m}} \approx 700$
- Comparing cross-section areas: $\dfrac{\pi(5\,\text{mm})^2}{\pi(7.5\,\mu\text{m})^2} \approx 450,000$

Image from The McGill Physiology Virtual Lab

`http://www.medicine.mcgill.ca/physio/vlab/cap/nerve_anat.htm`

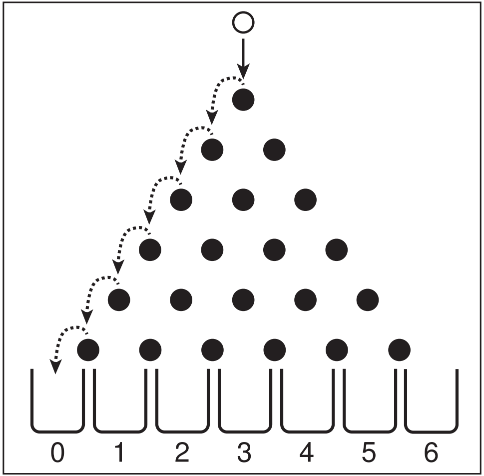
The Six-row Plinko



Bucket No.	Paths
0	
1	
2	
3	
4	
5	
6	

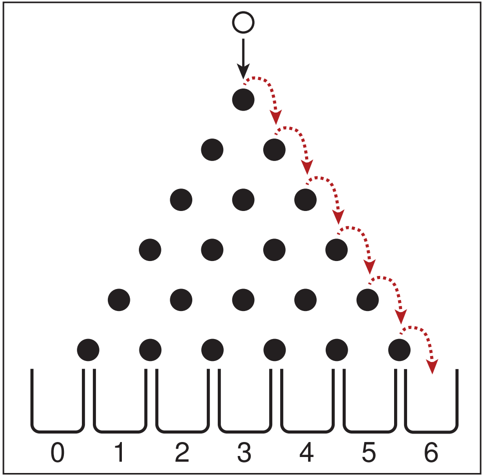
- $2^6 = 64$ total paths to the 7 buckets.

One Path to Bucket 0



Bucket No.	Paths
0	1
1	
2	
3	
4	
5	
6	

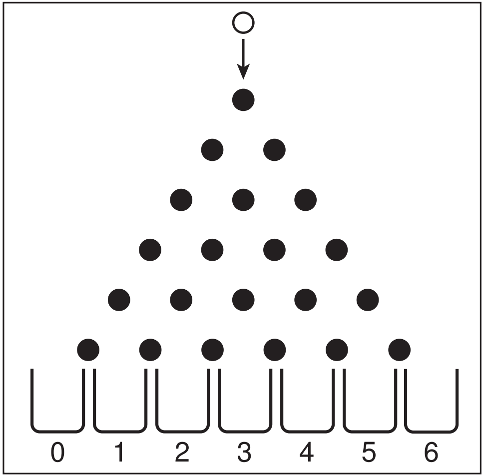
One Path to Bucket 6



Bucket No.	Paths
0	1
1	
2	
3	
4	
5	
6	1

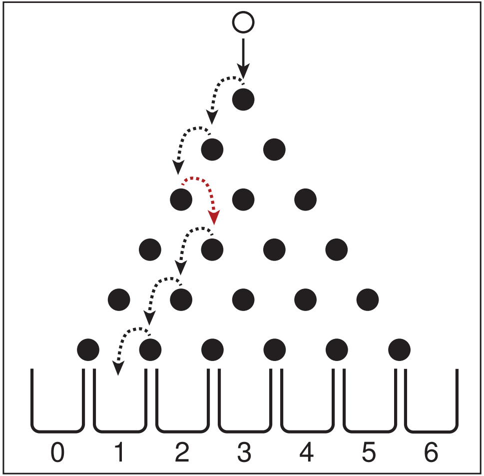
Clicker Question #1

How many paths are there to bucket 1?



1. 1
2. 2
3. 4
4. 6
5. 15
6. 30

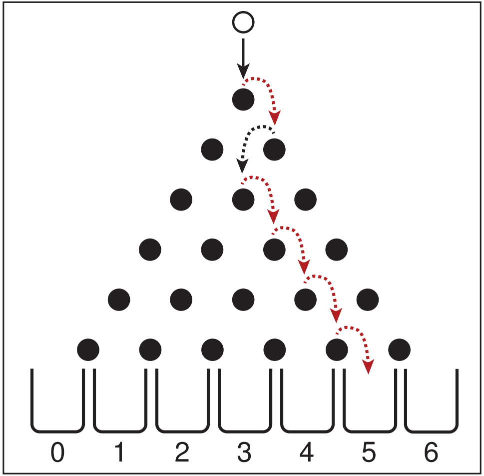
6 Paths to Bucket 1



Bucket No.	Paths
0	1
1	6
2	
3	
4	
5	
6	1

- Each path to bucket 1 includes 1 turn to the right and 5 to the left.

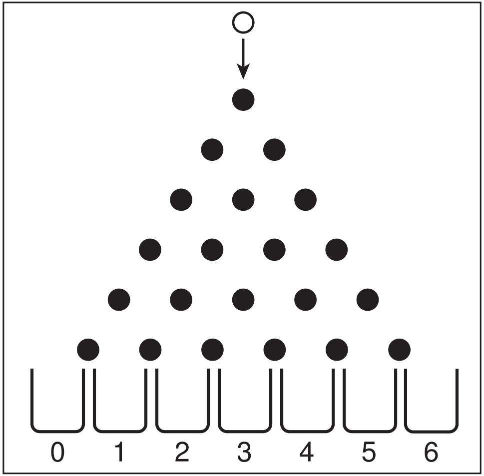
6 Paths to Bucket 5



Bucket No.	Paths
0	1
1	6
2	
3	
4	
5	6
6	1

- Each path to bucket 5 includes 5 turns to the right and 1 to the left.

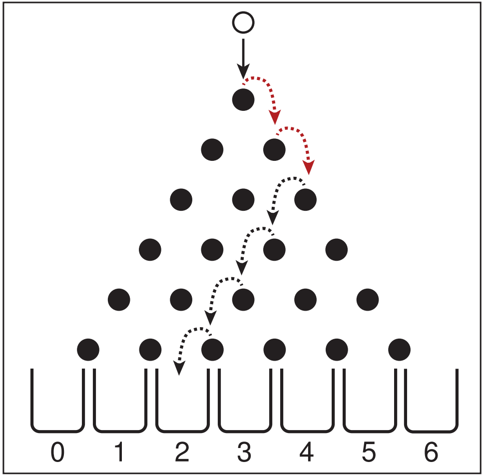
How Many Paths to Bucket 2?



1st right turn row	Paths
1	
2	
3	
4	
5	
6	

- Each path to bucket 2 includes 2 turns to the right and 4 to the left.

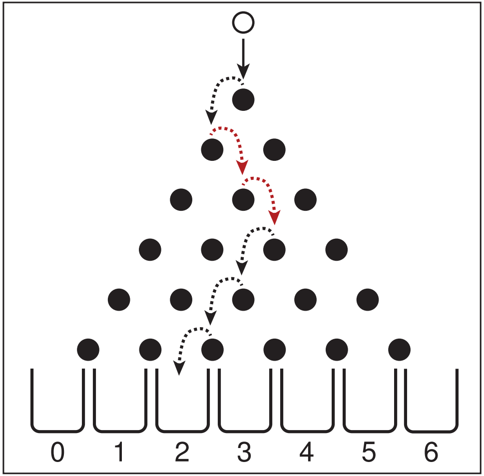
How Many Paths to Bucket 2?



1st right turn row	Paths
1	5
2	
3	
4	
5	
6	

- Each path to bucket 2 includes 2 turns to the right and 4 to the left.

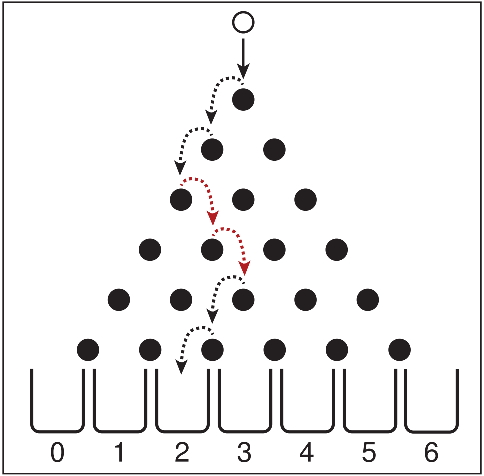
How Many Paths to Bucket 2?



1st right turn row	Paths
1	5
2	4
3	
4	
5	
6	

- Each path to bucket 2 includes 2 turns to the right and 4 to the left.

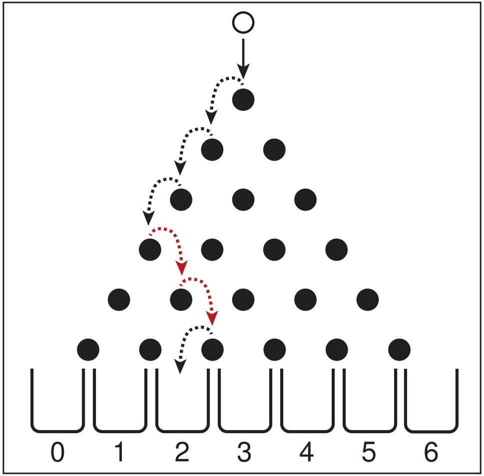
How Many Paths to Bucket 2?



1^{st} right turn row	Paths
1	5
2	4
3	3
4	
5	
6	

- Each path to bucket 2 includes 2 turns to the right and 4 to the left.

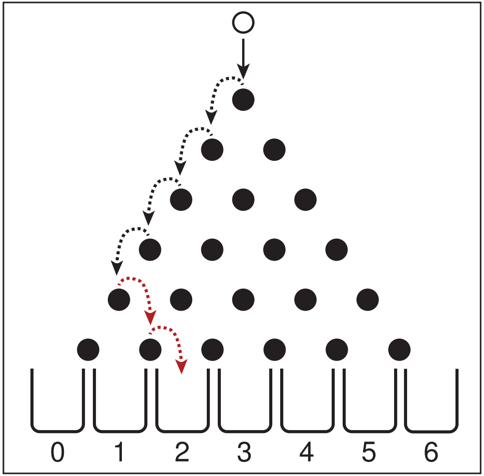
How Many Paths to Bucket 2?



1^{st} right turn row	Paths
1	5
2	4
3	3
4	2
5	
6	

- Each path to bucket 2 includes 2 turns to the right and 4 to the left.

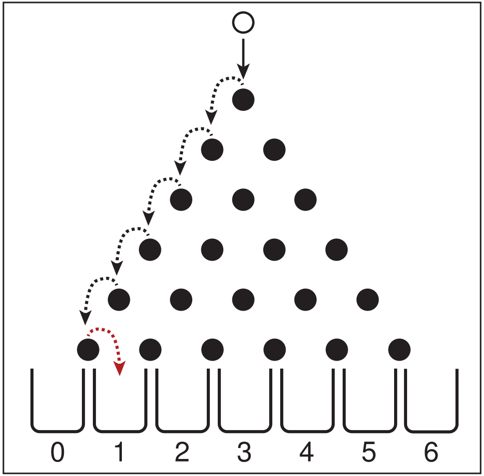
How Many Paths to Bucket 2?



1st right turn row	Paths
1	5
2	4
3	3
4	2
5	1
6	

- Each path to bucket 2 includes 2 turns to the right and 4 to the left.

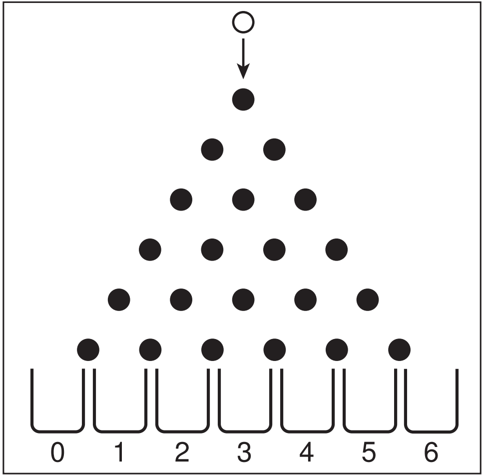
How Many Paths to Bucket 2?



1^{st} right turn row	Paths
1	5
2	4
3	3
4	2
5	1
6	0

- Each path to bucket 2 includes 2 turns to the right and 4 to the left.

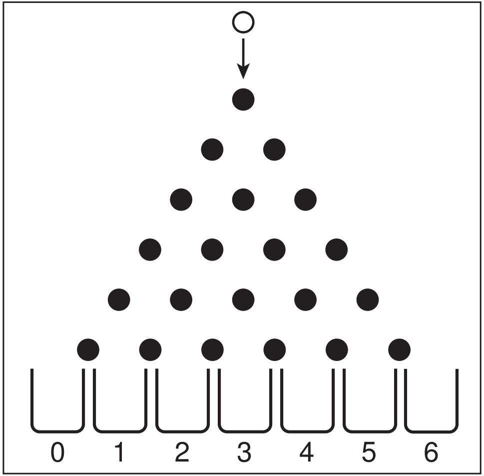
15 Paths to Bucket 2



Bucket No.	Paths
0	1
1	6
2	15
3	
4	
5	6
6	1

- Each path to bucket 2 includes 2 turns to the right and 4 to the left.

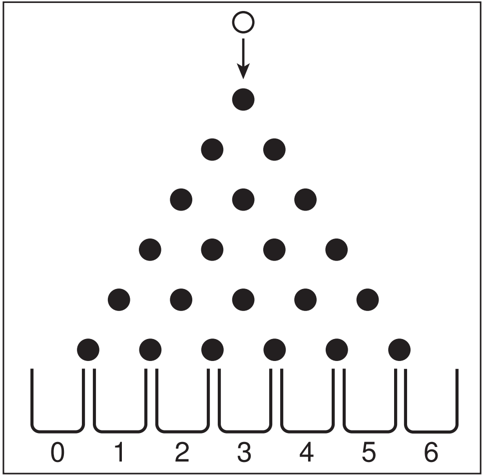
15 Paths to Bucket 4



Bucket No.	Paths
0	1
1	6
2	15
3	
4	15
5	6
6	1

- Each path to bucket 4 includes 4 turns to the right and 2 to the left.

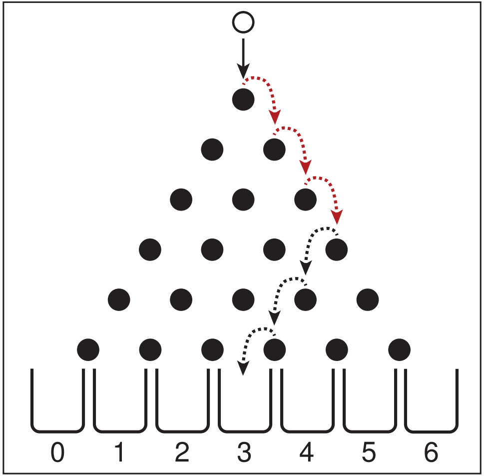
How Many Paths to Bucket 3



Bucket No.	Paths
0	1
1	6
2	15
3	
4	15
5	6
6	1

- Each path to bucket 3 includes 3 turns to the right and 3 to the left.

How Many Paths to Bucket 3



Bucket No.	Paths
0	1
1	6
2	15
3	
4	15
5	6
6	1

- Each path to bucket 3 includes 3 turns to the right and 3 to the left.

Another Way to Count the Paths to Bucket 2

- 2 turns to the right and 4 turns to the left.
- There are 6 rows where one turn could be placed.
- There are 5 rows where a second turn could be placed.
  $6 \times 5 = 30$
- BUT, this assumes that the turns can be placed in either order!
- The first turn has to come before the second, so that each possible path has been counted twice.
- The correct count: $6 \times 5 \times \frac{1}{2} = 15$
- A general strategy: Count all of the possible places where each right turn could be placed, allowing all possible orders, and then correct for over counting.

Counting the Paths to Bucket 3

- 3 turns to the right and 3 to the left.
- Ignoring the order of placement:
  - 6 rows where a first turn can be placed.
  - 5 rows where a second turn can be placed.
  - 4 rows where a third turn can be placed.
  - $6 \times 5 \times 4 = 120$
- But, this assumes turns can be placed in any order!
- How do we determine how many times the paths have been over counted?

A Related Problem: Placing Beans in Cups

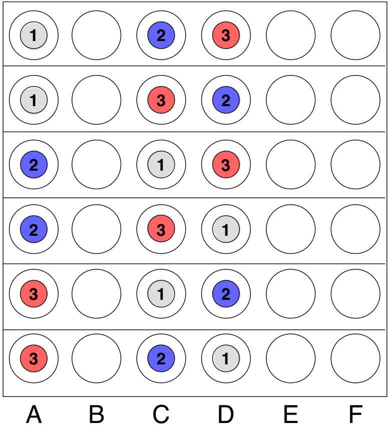
- Suppose that we have 3 beans, each with the number 1, 2 or 3 printed on it.
- How many different ways are there to place the beans in 6 cups?
  - 6 cups where the first bean can be placed.
  - 5 cups where the second bean can be placed.
  - 4 cups where the third bean can be placed.
  - $6 \times 5 \times 4 = 120$
- These are all different, because the beans are distinguishable.

Clicker Question #2

How many ways are there to put 3 labeled beans in 3 specific cups?

1. 1
2. 3
3. 6
4. 9
5. 12

Three Labeled Beans in Three Cups



- How many ways are there to put 3 labeled beans in 3 **specific** cups?
  $3 \times 2 \times 1 = 6$
- Only one of these has the order 1-2-3.

Three Labeled Beans in Six Cups

- If all orders are counted:

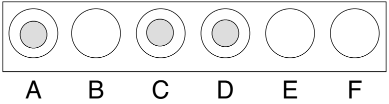
  Number of ways to place beans is: $6 \times 5 \times 4 = 120$

- If only the placements with the order 1-2-3 are counted:

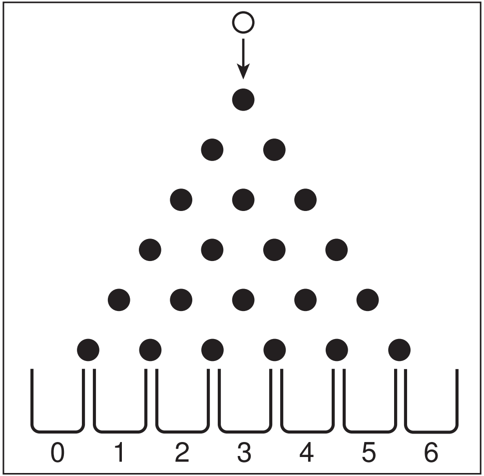
  Number of ways to place beans is: $(6 \times 5 \times 4) \div 6 = 20$

- Also the number of ways to place 3 turns to the right in the 6-row plinko!

- Also the number of ways to place 3 indistinguishable beans in 6 cups.

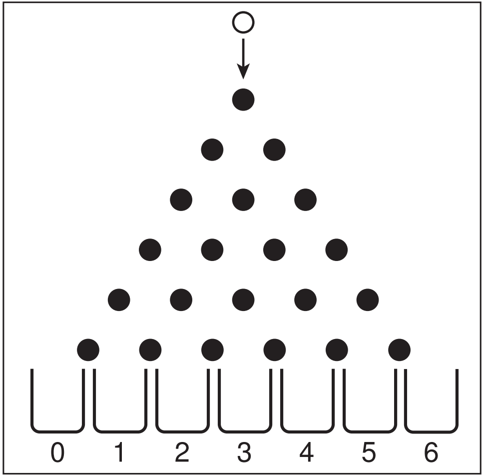


The Full Path Count for the Six-row Plinko



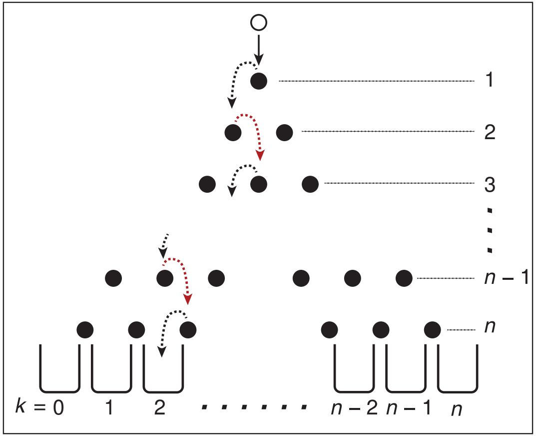
Bucket No.	Paths
0	1
1	6
2	15
3	20
4	15
5	6
6	1

Probabilities for the Six-row Plinko



Bucket No.	Paths	Probability
0	1	$1/64 \approx 0.016$
1	6	$6/64 \approx 0.094$
2	15	$15/64 \approx 0.234$
3	20	$20/64 \approx 0.312$
4	15	$15/64 \approx 0.234$
5	6	$6/64 \approx 0.094$
6	1	$1/64 \approx 0.016$

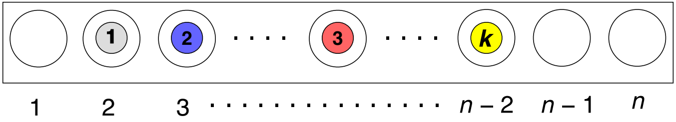
An n-row Plinko



- $k =$ bucket number.
- To reach bucket k, ball must make k turns to the right and $n - k$ turns to the left.

Beans and Cups

- For an n-row plinko, the number of paths to bucket k is the number of ways to place k labeled beans in n cups **in a single order**.



- To calculate this number:
  1. Calculate the number of ways to place k labeled beans in n cups, in any order.
  2. Calculate the number of ways to place k labeled beans in k cups, in any order.
  3. Divide result of 1 by result of 2.

Combinations, Permutations and the Factorial Function

- Combination: Selection of k objects from a set of n, without duplication and without regard to order.
- Permutation: Specific ordering of a set of distinguishable objects.
  - The number of ways to place k labeled beans in k cups, $P(k)$.

$$
\begin{aligned}
P(1) &= 1 \\
P(2) &= 2 \cdot 1 = 1 \\
P(3) &= 3 \cdot 2 \cdot 1 = 6 \\
P(k) &= n(n-1)(n-2) \cdots 2 \cdot 1
\end{aligned}
$$

(P for permutation, p for probability)

- The factorial function for positive integers:

$$
\begin{aligned}
k! &= k(k-1)(k-2) \cdots 2 \cdot 1 \\
0! &= 1 \\
P(k) &= k!
\end{aligned}
$$

A Special Kind of Permutation

- $P(k, n)$: The number of ways to choose k distinguishable objects from a set of n, in any order.

$$P(2,3) = 3 \cdot 2$$
$$P(3,5) = 5 \cdot 4 \cdot 3$$
$$P(k,n) = n(n-1)(n-2)\cdots(n-k+1)$$

$$P(k,n) = \frac{n(n-1)\cdots(n-k+1)(n-k)(n-k-1)\cdots 2 \cdot 1}{(n-k)(n-k-1)\cdots 2 \cdot 1} = \frac{n!}{(n-k)!}$$

- $P(k, n)$ is also the number of different (ordered) ways to put k labeled beans in n cups.

Back to the Plinko

- For an n-row plinko, the number of paths to bucket k is the number of ways to place k labeled beans in n cups **in a single order**.
- The number of ordered ways to put k labeled beans in n cups is:

$$P(k, n) = n(n-1)(n-2)\cdots(n-k+1) = \frac{n!}{(n-k)!}$$

- The number of ordered ways to place k labeled beans in k cups is:

$$P(k) = k!$$

- The number of ways to place k labeled beans in n cups **in a single order** is:

$$\frac{P(k, n)}{P(k)} = \frac{n!}{k!(n-k)!}$$

Test Recipe on the Six-Row Plinko

- Paths to bucket 0: $\frac{n!}{k!(n-k)!} = \frac{6!}{0!6!} = \frac{720}{1 \cdot 720} = 1$
- Paths to bucket 1: $\frac{n!}{k!(n-k)!} = \frac{6!}{1!5!} = \frac{720}{1 \cdot 120} = 6$
- Paths to bucket 2: $\frac{n!}{k!(n-k)!} = \frac{6!}{2!4!} = \frac{720}{2 \cdot 24} = 15$
- Paths to bucket 3: $\frac{n!}{k!(n-k)!} = \frac{6!}{3!3!} = \frac{720}{6 \cdot 6} = 20$

Test Recipe on the Six-Row Plinko (contd.)

- Paths to bucket 4: $\dfrac{n!}{k!(n-k)!} = \dfrac{6!}{4!2!} = \dfrac{720}{24 \cdot 2} = 15$
- Paths to bucket 5: $\dfrac{n!}{k!(n-k)!} = \dfrac{6!}{5!1!} = \dfrac{720}{120 \cdot 1} = 6$
- Paths to bucket 6: $\dfrac{n!}{k!(n-k)!} = \dfrac{6!}{6!0!} = \dfrac{720}{720 \cdot 1} = 1$
- It works!
  Now we can do any size plinko!

n choose k

- The expression we just derived applies to much more than plinkos!
- The expression is often written as:

$$\binom{n}{k} = \frac{n!}{(n-k)!k!}$$

and spoken as "n choose k"

- From n objects, choose k of them (each only once) and either
  - Only a single order is valid (*e.g.* turns in the plinko)

    Or
  - The order doesn't matter (*e.g.* unlabeled beans).

Clicker Question #3

For a 5-row plinko, with 6 buckets labeled 0 to 5, how many paths are there to bucket 3?

1. 2
2. 4
3. 6
4. 8
5. 10

$$\binom{5}{3} = \frac{5!}{3!(5-3)!} = \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{(3 \cdot 2 \cdot 1)(2 \cdot 1)} = \frac{120}{12} = 10$$

Binomial Coefficients

- The series of numbers generated by

$$\binom{n}{k} = \frac{n!}{(n-k)!k!}$$

  for a specific value of n and increasing values of $k \leq n$ are called "binomial coefficients."

- The binomial coefficients arise in algebra in the expansion of a sum of two terms:

$$(a+b)^2 = a^2 + 2ab + b^2$$
$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$
$$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

$$(a+b)^6 = a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$$