Physical Principles in Biology
Biology 3550
Fall 2016

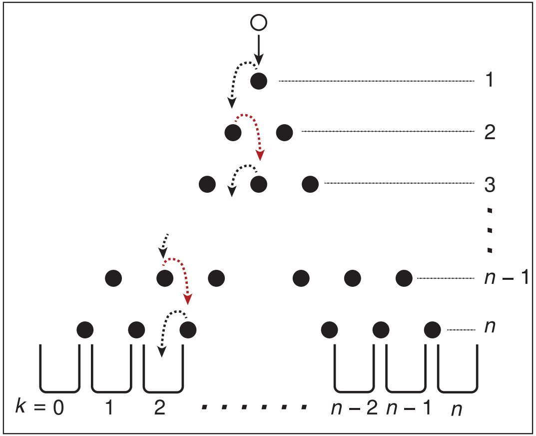
Lecture 8

The Binomial Probability Distribution and Expected Values

Friday, 9 September

University of Utah
goldenberg@biology.utah.edu

An n-row Plinko



- $k =$ bucket number.
- To reach bucket k, ball must make k turns to the right and $n - k$ turns to the left.

Counting Paths for the n-row Plinko

For an n-row plinko, the number of paths to bucket k is calculated as:

$$\binom{n}{k} = \frac{P(k,n)}{P(k)} = \frac{n!}{(n-k)!} \cdot \frac{1}{k!}$$

- "n choose k" is the number of ways to place k labeled beans in n cups **in a single order**.
- "n choose k" is also the number of ways to place k indistinguishable beans in n cups.
- The permutation, $P(k)$, is the number of ways to place k distinguishable items in different orders. (*e.g.* k labeled beans in k cups)
- The permutation $P(k,n)$ is the number of ways of choosing k distinguishable items from a total of n objects in different orders. (*e.g.* k labeled beans in n cups)

Probabilities for an n-row Plinko

- The total number of paths is 2^n.
- If each turn to the right or left is equally probable, the probabilities of all paths are equal, and the probability of each path is:

$$p = \frac{1}{2^n} = 2^{-n}$$

- The probability of a ball landing in bucket k is the number of paths to the bucket multiplied by the probability of each path:

$$p(k) = \frac{n!}{k!(n-k)!} \cdot 2^{-n}$$

Clicker Question #1

For a 7-row plinko, with 8 buckets labeled 0 to 7, what is the probability of a ball landing in bucket 1?

(There's a hard way and an easy way!)

1. ~ 0.01
2. ~ 0.05
3. ~ 0.1
4. ~ 0.15
5. ~ 0.2

$$p(1) = \frac{7!}{1!(n-1)!} \cdot 2^{-7} = \frac{7!}{6!} \cdot 2^{-7} = 7 \cdot 2^{-7}$$

What if the Plinko is Biased?

- Suppose that each peg in the plinko has been “fixed”, so that the probability of a left turn is 0.4 and the probability of a right turn is 0.6.
- For each of the paths to bucket k, there are k right turns and $(n - k)$ left turns.
- For each individual path to bucket k, the probability is:

$$0.6^k \times 0.4^{(n-k)}$$

- The total probability of a ball falling in bucket k is:

$$p(k) = \frac{n!}{k!(n-k)!} \times 0.6^k \times 0.4^{(n-k)}$$

For a Biased Six-Row Plinko

Bucket	Paths	Probability $p(\text{right}) = 0.5$	Probability $p(\text{right}) = 0.6$
0	1	~ 0.016	~ 0.004
1	6	~ 0.094	~ 0.037
2	15	~ 0.234	~ 0.138
3	20	~ 0.312	~ 0.276
4	15	~ 0.234	~ 0.311
5	6	~ 0.094	~ 0.187
6	1	~ 0.016	~ 0.047

The Binomial Probability Distribution Function

- The general formulation:
  $p(k; n, p)$ is the probability of k successes in n successive binary (yes/no) trials when the probability of success in each trial is p.
- The probability function:

$$p(k; n, p) = \frac{n!}{k!(n-k)!} p^k (1-p)^{(n-k)}$$

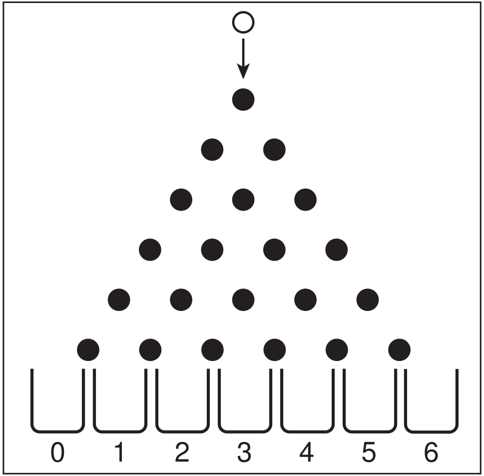
- Some applications beyond plinkos:
  - Number of heads in n successive coin tosses.
  - Number of successes in prescribing a medication to a series of patients with the same condition.
  - Probability of surviving n potentially deadly events.
    $p(n; n, p)$, were p is the probability of surviving each event

Playing Plinko for Cash

- Suppose that I let you put a ball in the 6-row plinko, and I agree to pay you k dollars if the ball lands in bucket k.
- This is probably going to cost me money!
- How much should I charge you to play?
- How much, on average, am I going to have to pay?

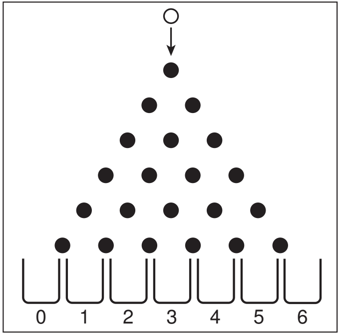
Clicker Question #2

How much should I charge you to play my plinko game (to break even)?



1. $1
2. $2
3. $3
4. $4
5. $6
6. $7

An Intuitive Solution



- Buckets 0 and 6 have equal probabilities. The average payout for these two is $3.
- Buckets 1 and 5 have equal probabilities. The average payout for these two is $3.
- Buckets 2 and 4 have equal probabilities. The average payout for these two is $3.
- The payout for bucket 3 is $3.
- The overall average payout is $3.

Random Variables

- Definition: A variable that is assigned a value for each possible outcome or event for a probabilistic process.
- Examples:
  - For a coin toss, we could assign a random variable, x, the value of 1 for heads or 0 for tails.
  - For n successive coin tosses, we could define x to be the number of heads.
  - For the Plinko, we can define the random variable, x, as the number of the bucket that the ball lands in.
    But, we could define other random variables, too.

The Expected Value or Expectation

For a random process that has n possible outcomes (or a complete set of n non-overlapping events):

- The random variable, x, has values of x_k for $k = 1, 2, 3 \ldots n$
- The n possible outcomes (or events) have probabilities of $p(k)$, for $k = 1, 2, 3 \ldots n$
- The expected value is defined as:

$$E = \sum_{k=1}^{n} p(k) x_k$$

- If the process is repeated a large number of times, the average value of x will approach E.
- For a game of chance, if x_k is the number of dollars paid out for outcome (or event), k, E is the average payout.

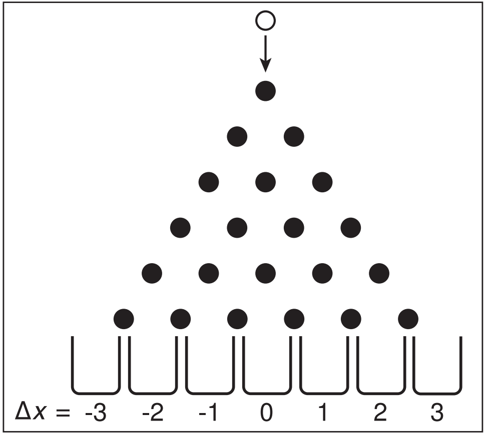
Expected Value for the Unbiased Six-row Plinko

Bucket	x	$p(x)$	$p(x)x$
0	0	1/64	0
1	1	6/64	6/64
2	2	15/64	30/64
3	3	20/64	60/64
4	4	15/64	60/64
5	5	6/64	30/64
6	6	1/64	6/64
Total		1	192/64 =3

Expected Value for a Biased Six-row Plinko: $p(\text{right}) = 0.6$

Bucket	x	$p(x)$	$p(x)x$
0	1	0.004	0
1	1	0.037	0.037
2	2	0.138	0.276
3	3	0.276	0.829
4	4	0.311	1.244
5	5	0.186	0.933
6	6	0.046	0.280
Total		1	3.6

Another Random Variable for the Plinko, Δx



- Δx represents the position of the bucket, relative to the central bucket.

Expected Value of Δx for the Unbiased Six-row Plinko

Bucket	Δx	$p(\Delta x)$	$p(\Delta x)\Delta x$
0	-3	1/64	-3/64
1	-2	6/64	-12/64
2	-1	15/64	-15/64
3	0	20/64	0
4	1	15/64	15/64
5	2	6/64	12/64
6	3	1/64	3/64
Total		1	0

Expected Value of Δx for a Biased Six-row Plinko: $p(\text{right}) = 0.6$

Bucket	Δx	$p(\Delta x)$	$p(\Delta x)\Delta x$
0	-3	0.004	-0.012
1	-2	0.037	-0.074
2	-1	0.138	-0.138
3	0	0.276	0
4	1	0.311	0.311
5	2	0.186	0.373
6	3	0.046	0.139
Total		1	0.6

Notice:

- For the unbiased six-row plinko:

$$E(x) = 3$$
$$E(\Delta x) = 0$$

- For the biased six-row plinko:

$$E(x) = 3.6$$
$$E(\Delta x) = 0.6$$

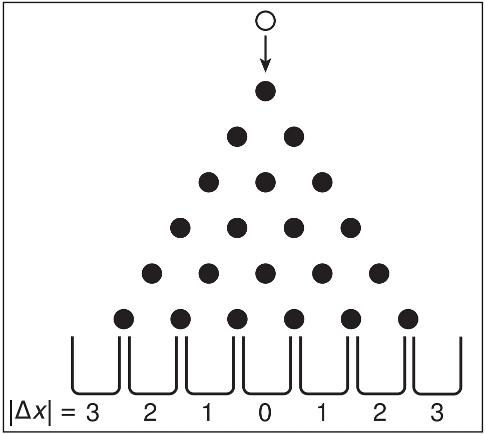
- For both, $\Delta x = x - 3$
- In general, if x is a random variable, and a is a constant:

$$E(a + x) = a + E(x)$$

- Also:

$$E(ax) = aE(x)$$

Another Random Variable for the Plinko, $|\Delta x|$



- $|\Delta x|$ represents the average distance from the central bucket.

Expected Value of $|\Delta x|$ for the Unbiased Six-row Plinko

Bucket	$\|\Delta x\|$	$p(\|\Delta x\|)$	$p(\|\Delta x\|)\,\|\Delta x\|$
0	3	1/64	3/64
1	2	6/64	12/64
2	1	15/64	15/64
3	0	20/64	0
4	1	15/64	15/64
5	2	6/64	12/64
6	3	1/64	3/64
Total		1	$60/64 \approx 0.94$

Expected Value of $|\Delta x|$ for a Biased Six-row Plinko: $p(\text{right}) = 0.6$

Bucket	Δx	$p(\Delta x)$	$p(\Delta x)\Delta x$
0	3	0.004	0.012
1	2	0.037	0.074
2	1	0.138	0.138
3	0	0.276	0
4	1	0.311	0.311
5	2	0.186	0.373
6	3	0.046	0.139
Total		1	1.05