Physical Principles in Biology
Biology 3550
Fall 2016

Lecture 9

Random Walks in One Dimension

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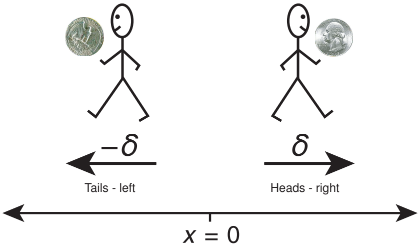
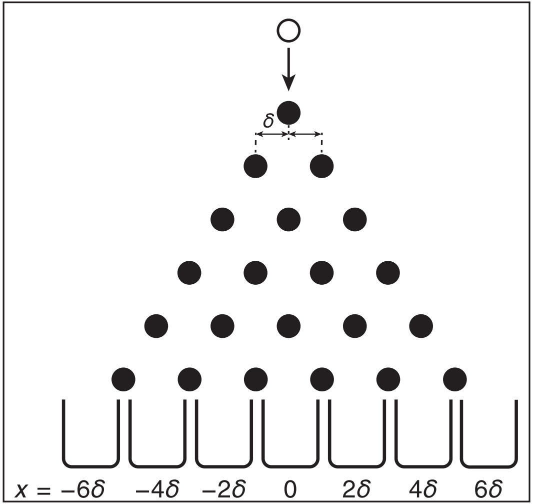
A Random Walk in One Dimension



1. Start at position $x = 0$.
2. Flip coin.
   - Heads, take step of length δ to the right.
   - Tails, take step of length δ to the right.
3. Repeat 2 another $(n - 1)$ times.
4. Final position is $x(n)$.

Like a Plinko, with variable x



- x represents the position of the bucket, relative to the central bucket.

What Do We Know About $x(n)$, the End Point?

- The maximum value of $x(n)$ is δn.
- The minimum value of $x(n)$ is $-\delta n$.
- If we repeat the random walk many times:
  - The distribution of $x(n)$ will be binomial.
  - The average value of $x(n)$ will approach 0. ($E(x) = 0$))
    (if the coin is fair)
- But, if n is very large, calculating the distribution will be difficult!

Another Way to Calculate The Average Final Position (The Expected Value of $x(n)$)

- For a single random walk, the final position will be:

$$x(n) = \sum_{i=1}^{n} \delta_i$$

  where i is the step number, and δ_i is either $-\delta$ or $+\delta$, with equal probability (if the coin is fair), for each step.

- For each step, δ_i is a random variable, with an expected value, $E(\delta_i)$. If the coin is fair, the expected value of δ_i is zero:

$$E(\delta_i) = p(+\delta)\delta + p(-\delta)(-\delta)$$

$$= 0.5\delta - 0.5\delta = 0$$

Calculating The Expected Value of $x(n)$

- An important theorem: If x and y are two independent random variables, then the expected value of the sum is calculated as:

$$E(x+y) = E(x) + E(y)$$

- Since:

$$x(n) = \sum_{i=1}^{n} \delta_i$$

The expected value of $x(n)$ is calculated as:

$$E\big(x(n)\big) = \sum_{i=1}^{n} E(\delta_i)$$

- If $E(\delta_i) = 0$ for each step, then $E\big(x(n)\big) = 0$

If the Coin is Biased

- Define probability of heads ($\delta_i = +\delta$) for step i as $p(+\delta)$, and probability of tails ($\delta_i = -\delta$) for step i as $p(-\delta)$

$$p(-\delta) = 1 - p(+\delta)$$

- The expected value of δ_i:

$$\begin{aligned}
E(\delta_i) &= p(+\delta)\delta + p(-\delta)(-\delta) \\
&= p(+\delta)\delta - \big(1 - p(+\delta)\big)\delta \\
&= \delta\Big(p(+\delta) - \big(1 - p(+\delta)\big)\Big) \\
&= \delta\big(p(+\delta) - 1 + p(+\delta)\big) \\
&= \delta\big(2p(+\delta) - 1\big)
\end{aligned}$$

Some Examples for $E(\delta_i)$

$$E(\delta_i) = \delta\big(2p(+\delta) - 1\big)$$

$p(+\delta)$	$E(\delta_i)$
0.5	0
1	$+\delta$
0	$-\delta$
0.6	0.2δ
0.4	-0.2δ

Clicker Question #1

For a coin that is biased so that $p\big((+\delta)\big) = 0.7$, and $+\delta = 2$ what is $E(\delta_i)$?

1. 0.7
2. 0.8
3. 0.9
4. 1.0
5. 1.2
6. 1.4

Calculating The Expected Value of $x(n)$, with a biased coin

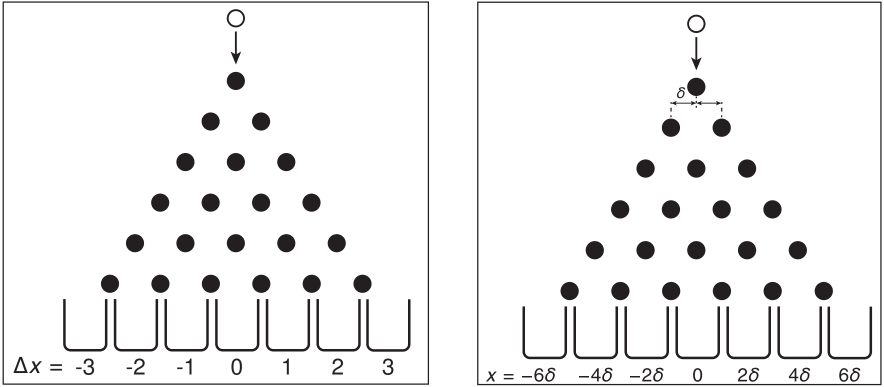
- The expected value of $x(n)$ is calculated as:

$$E\big(x(n)\big) = \sum_{i=1}^{n} E(\delta_i)$$

- If $E(\delta_i)$ has the same value for all steps:

$$\begin{aligned} E\big(x(n)\big) &= nE(\delta_i) \\ &= n\delta\big(2p(+\delta) - 1\big) \end{aligned}$$

Revisiting The Six-row Plinko



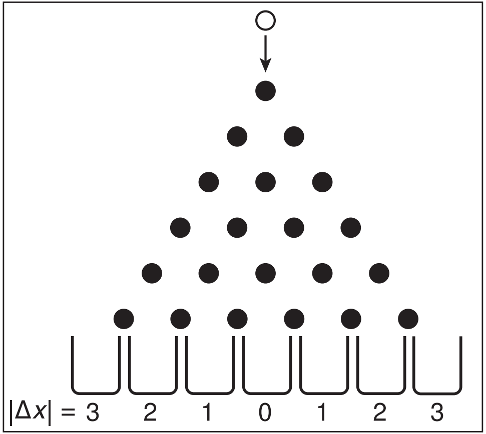
In today's notation

- $x = \Delta x$, $\delta = 1/2$, n=6
- $p(+\delta) = 0.6$, $E(\delta_i) = \delta\big(2p(+\delta) - 1\big) = 0.1$
- $E(x) = nE(\delta_i) = 0.6$

Expected Value of Δx for a Biased Six-row Plinko: $p(\text{right}) = 0.6$

Bucket	Δx	$p(\Delta x)$	$p(\Delta x)\Delta x$
0	-3	0.004	-0.012
1	-2	0.037	-0.074
2	-1	0.138	-0.138
3	0	0.276	0
4	1	0.311	0.311
5	2	0.186	0.373
6	3	0.046	0.139
Total		1	0.6

Another Random Variable for the Plinko, $|\Delta x|$



- $|\Delta x|$ represents the average distance from the central bucket.
- Calculating the expected value of $|\Delta x|$ is not as simple as adding up $E(|\delta_i|)$.

Some Different Kinds of Average

For N random walks of n steps each:

- The mean:

$$\langle x(n) \rangle = \frac{1}{N} \sum_{j=1}^{N} x_i(n), \text{ approaches } E\big(x(n)\big)$$

- The mean-square:

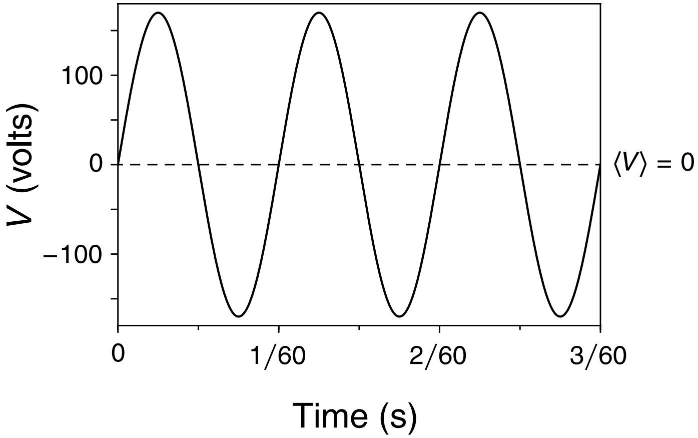
$$\langle x(n)^2 \rangle = \frac{1}{N} \sum_{j=1}^{N} x_i(n)^2$$

- The root-mean-square (RMS):

$$\mathrm{RMS}(x(n)) = \sqrt{\langle x(n)^2 \rangle} = \sqrt{\frac{1}{N} \sum_{j=1}^{N} x_i(n)^2}$$

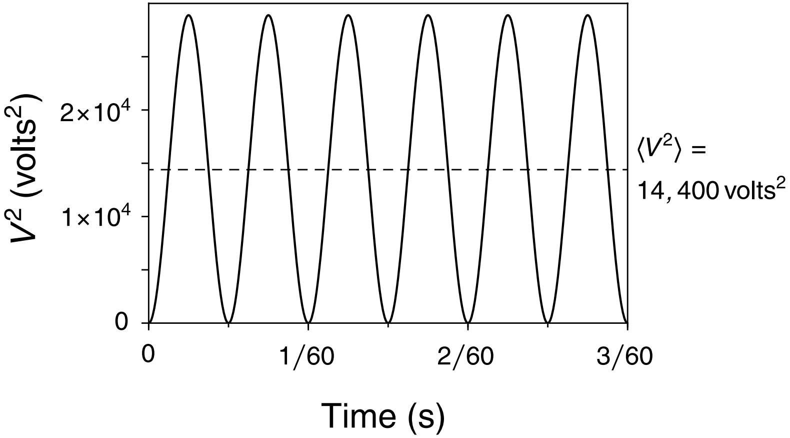
An Application of Mean-square and Root-mean-square Averages: Household Power (US)

Voltage versus time



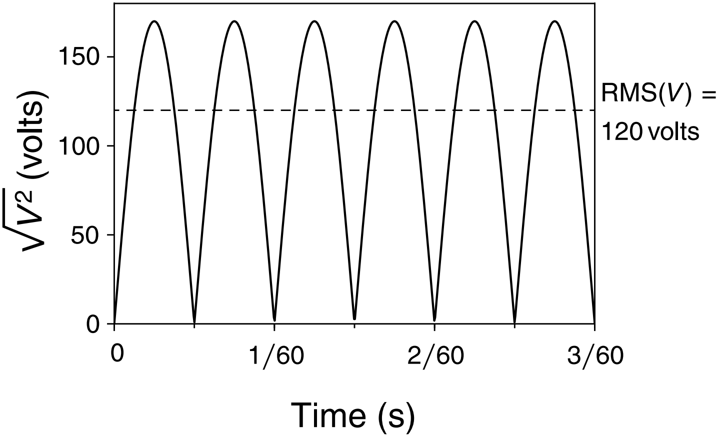
An Application of Mean-square and Root-mean-square Averages: Household Power (US)

Voltage squared versus time



An Application of Mean-square and Root-mean-square Averages: Household Power (US)

RMS Voltage versus time



Clicker Question #2

For the numbers: $-4, 2, -3, 1, 5,$
Calculate the root-mean-square average

1. ~ 0.2
2. ~ 1.5
3. ~ 2.9
4. ~ 3.3
5. ~ 4.8

$$\text{RMS} = \sqrt{\frac{-4^2 + 2^2 + -3^2 + 1^2 + 5^2}{5}} = \sqrt{\frac{16 + 4 + 1 + 25}{5}} = \sqrt{\frac{55}{5}} = \sqrt{11}$$

Calculating the Mean-Square Displacement for a 1-d Random Walk

- For a single random walk, the final position will be:

$$x(n) = \sum_{i=1}^{n} \delta_i$$

where i is the step number, and δ_i is either $-\delta$ or $+\delta$, with equal probability (if the coin is fair), for each step.

- We can also express $x(n)$ in terms of the position after the next-to-last step, $x(n-1)$:

$$x(n) = x(n-1) + \delta_n$$

Calculating The Mean-Square Displacement

- If we do a large number, N, of random walks, the mean-square displacement, $\langle x \rangle$, will be:

$$\langle x(n)^2 \rangle = \frac{1}{N} \sum_{j=1}^{N} x_j(n)^2 = \frac{1}{N} \sum_{j=1}^{N} \left(\sum_{i=1}^{n} \delta_{j,i} \right)^2$$

where j is the random walk number, and $\delta_{j,i}$ is the displacement for step i of walk j.

- We can also write the mean-square average as:

$$\langle x(n)^2 \rangle = \frac{1}{N} \sum_{j=1}^{N} \left(x_j(n-1) + \delta_{j,n} \right)^2$$

$$= \frac{1}{N} \sum_{j=1}^{N} \left(x_j(n-1)^2 + 2x_j(n-1)\delta_{j,n} + \delta_{j,n}^2 \right)$$

$\delta_{n,j}$ is the change in position for the last step in random walk j.

Calculating The Mean-Square Displacement

- Following from the previous slide:

$$\langle x(n)^2\rangle = \frac{1}{N}\sum_{j=1}^{N}\left(x_j(n-1)^2 + 2x_j(n-1)\delta_{j,n} + \delta_{j,n}^2\right)$$

$$= \frac{1}{N}\sum_{j=1}^{N} x_j(n-1)^2 + \frac{1}{N}\sum_{j=1}^{N}(2x_j(n-1)\delta_{j,n} + \frac{1}{N}\sum_{j=1}^{N}\delta_{j,n}^2$$

$$= \langle x(n-1)^2\rangle + \langle 2x(n-1)\delta_n\rangle + \langle\delta_n^2\rangle$$

- For each walk, δ_n is either $+\delta$ or $-\delta$, with equal probability (for an unbiased walk), and is independent of $2x(n-1)$.
- The average of $2x(n-1)\delta_n$ over N walks, $\langle 2x(n-1)\delta_n\rangle$, is expected to be 0.

Calculating The Mean-Square Displacement

- From the previous slide:

$$\langle x(n)^2 \rangle = \langle x(n-1)^2 \rangle + \langle \delta_n^2 \rangle$$

- Following the same logic:

$$\langle x(n-1)^2 \rangle = \langle x(n-2)^2 \rangle + \langle \delta_{n-1}^2 \rangle$$

- and

$$\begin{aligned} \langle x(n)^2 \rangle &= \langle x(n-2)^2 \rangle + \langle \delta_{n-1}^2 \rangle + \langle \delta_n^2 \rangle \\ &= \langle x(n-2)^2 \rangle + 2\langle \delta^2 \rangle \end{aligned}$$

Calculating The Mean-Square Displacement

- Continuing in the same way:

$$\begin{aligned}\langle x(n)^2\rangle &= \langle x(n-2)^2\rangle + 2\langle\delta^2\rangle \\ &= \langle x(n-3)^2\rangle + \langle\delta^2_{n-2}\rangle + 2\langle\delta^2\rangle \\ &= \langle x(n-3)^2\rangle + 3\langle\delta^2\rangle\end{aligned}$$

$$\langle x(n)^2\rangle = \langle x(n-4)^2\rangle + 4\langle\delta^2\rangle$$

- and so on, until we have:

$$\begin{aligned}\langle x(n)^2\rangle &= \langle x(1)^2\rangle + (n-1)\langle\delta^2\rangle \\ &= \langle x(0)^2\rangle + n\langle\delta^2\rangle \\ &= n\langle\delta^2\rangle\end{aligned}$$

The Mean-square Displacement for One Step: $\langle \delta^2 \rangle$

- For large N, $\langle \delta^2 \rangle \rightarrow E(\delta^2)$

$$\begin{aligned} E(\delta^2) &= p(+\delta)(+\delta)^2 + p(-\delta)(-\delta)^2 \\ &= p(+\delta)\delta^2 + p(-\delta)\delta^2 \\ &= \delta^2 \big(p(+\delta) + p(-\delta) \big) \\ &= \delta^2 \end{aligned}$$

- The mean-square displacement for the random walk:

$$\langle x(n)^2 \rangle = n\delta^2$$

- The root-mean-square displacement for the random walk:

$$\mathrm{RMS}(x(n)) = \sqrt{\langle x(n)^2 \rangle} = \sqrt{n}\delta$$

The Root-mean-square Displacement for a One-dimensional Random Walk

$$\mathrm{RMS}(x(n)) = \sqrt{\langle x(n)^2 \rangle} = \sqrt{n}\delta$$