Physical Principles in Biology
Biology 3550
Fall 2017

Lecture 11

Two-dimensional Random Walks

Friday, 15 September

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Announcements

- Quiz 2: Friday, 22 September
- Problem Set 2 now posted on lab web page
  Due Monday, 25 September
- Class session for Monday, 18 September:
  Special computer session to learn to use SciDAVis
  Room 150, South Biology building

Summary of Results for One-dimensional Random Walks (with notation change: $x(n) = x_n$)

Averages for a large number, N of random walks of n steps each, with step length δ:

- The mean:

$$\langle x_n \rangle = n\delta\big(2p(+\delta) - 1\big)$$

where $p(+\delta)$ is the probability of a forward step.

- The mean-square:

$$\langle x_n^2 \rangle = n\delta^2$$

Assumes $\langle x_n \rangle = 0$ and $\langle \delta_i^2 \rangle = \delta^2$, where δ_i is the change in position in step i.

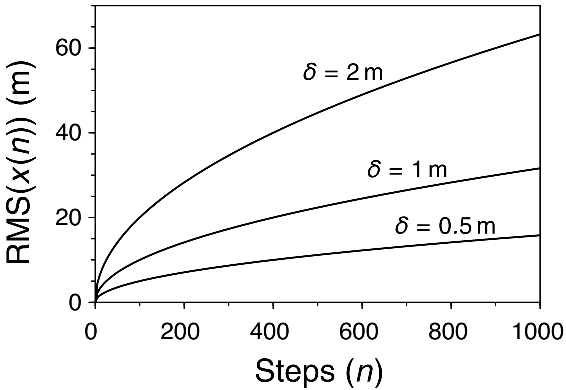
- The root-mean-square (RMS):

$$\mathrm{RMS}(x_n) = \sqrt{\langle x_n^2 \rangle} = \sqrt{n}\delta$$

with the same assumptions as for $\langle x_n^2 \rangle$.

The Root-mean-square Displacement for a One-dimensional Random Walk

$$\mathrm{RMS}(x_n) = \sqrt{\langle x_n^2 \rangle} = \sqrt{n}\delta$$



- This is THE most important thing to remember about random walks!

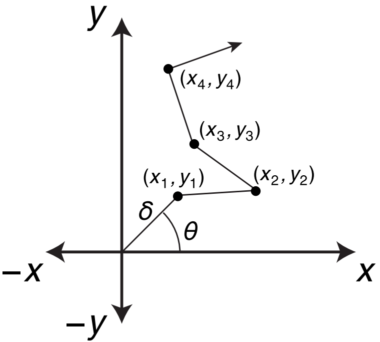
Clicker Question #1

For a one-dimensional random walk of 10 steps, of length 2.5 m, what is the expected RMS distance between the starting and ending points?

1. 2.5 m
2. 5 m
3. 7.9 m
4. 25 m
5. 50 m
6. 79 m

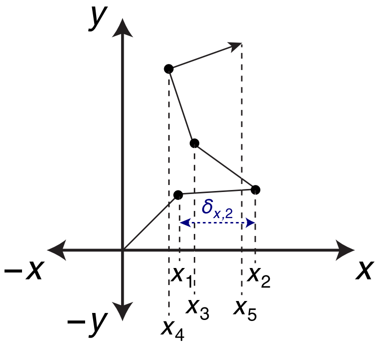
$$\mathrm{RMS}(x_n) = \sqrt{n}\delta$$
$$= \sqrt{10} \times 2.5\,\mathrm{m} \approx 3.162 \times 2.5\,\mathrm{m} \approx 7.9\,\mathrm{m}$$

A Random Walk in Two Dimensions



1. Start at (x, y) coordinates (0,0).
2. Choose a random direction, defined by the angle θ from the x-axis.
3. Move distance δ in the chosen direction.
4. Repeat 2 and 3 another $(n - 1)$ times.

A Random Walk in Two Dimensions



- x-coordinates represent a random walk along the x-axis.
- Can also describe a random walk along the y-axis (or any other direction).
- What are $\langle x_n \rangle$, $\langle x_n^2 \rangle$ and $\mathrm{RMS}(x_n)$?
- The change in x with each step, $\delta_{x,i}$ is not discrete!

A Brief Return to the Derivation of $\langle x_n^2 \rangle$ for a One-dimensional Random Walk

- Using a recursion argument, we showed that:

  $$\langle x_n^2 \rangle = n \langle \delta^2 \rangle$$

  This assumed only that $\langle x_n \rangle = 0$. (unbiased coin)

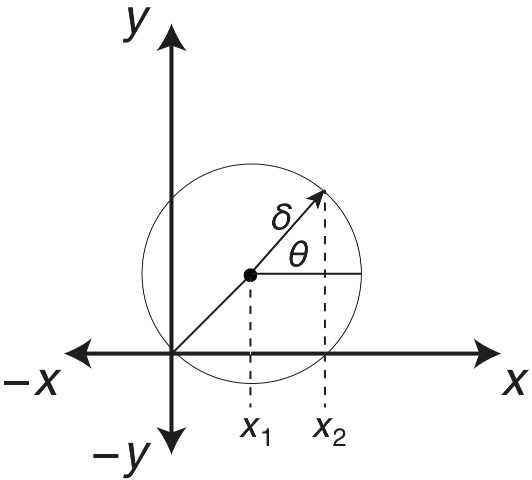
- A further assumption, that the individual displacements along the x-axis were either $+\delta$ or $-\delta$, allowed us to substitute δ^2 for $\langle \delta^2 \rangle$.

- For the two-dimensional random walk:
  - For the individual steps, a positive or negative displacement along the x-axis is equally likely, so the average, $\langle x_n \rangle$, should be zero. So, we can still write:

    $$\langle x_n^2 \rangle = n \langle \delta_x^2 \rangle$$

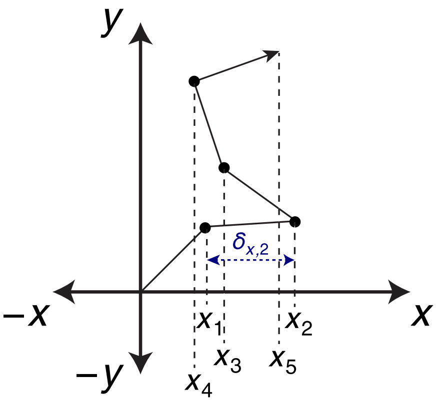
  - But, the displacements are not discrete, and $\langle \delta_x^2 \rangle$ is not equal to δ^2.

Why $\langle \delta_x^2 \rangle$ is not Equal to δ^2 for a Two-dimensional Random Walk



- For each random walk step:
  $\delta_{x,i} = x_i - x_{i-1}$
- If $\theta = 0$, then:
  $\delta_{x,i} = \delta$
  $\delta_{x,i}^2 = \delta^2$
- If $\theta = \pi$ rad, then:
  $\delta_{x,i} = -\delta$
  $\delta_{x,i}^2 = \delta^2$
- For most values of θ:
  $\left|\delta_{x,i}\right| < \delta$
  $\delta_{x,i}^2 < \delta^2$

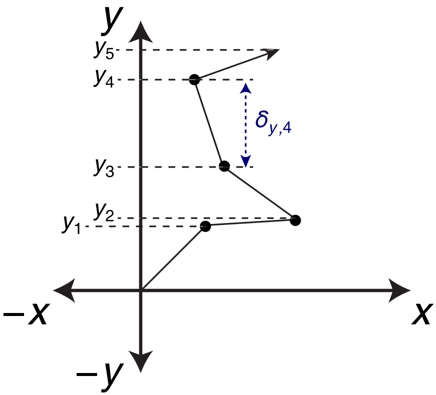
The Random Walk Along the x-axis



- $\langle x_n \rangle = 0$
- $\langle x_n^2 \rangle = n \langle \delta_x^2 \rangle$
- $\text{RMS}(x_n) = \sqrt{\langle x_n^2 \rangle} = \sqrt{n} \sqrt{\langle \delta_x^2 \rangle}$
- $\langle \delta_x^2 \rangle < \delta^2$
- RMS displacement along the x-axis is less than for a random walk constrained to the x-axis, with the same step size, δ.

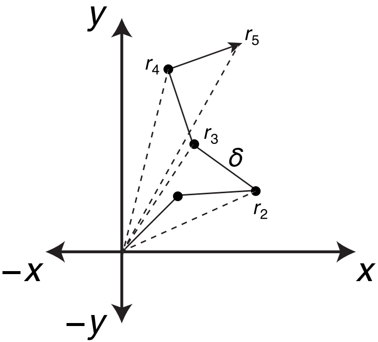
  By how much?

The Random Walk Along the y-axis



- $\langle y_n \rangle = 0$
- $\langle y_n^2 \rangle = n \langle \delta_y^2 \rangle$
- $\text{RMS}(x_n) = \sqrt{\langle x_n^2 \rangle} = \sqrt{n} \sqrt{\langle \delta_y^2 \rangle}$
- If all values of θ are equally probable:
  $\langle \delta_x^2 \rangle = \langle \delta_y^2 \rangle = \langle \delta_{x,y}^2 \rangle$
- The average x- and y-projections should be the same:
  $\langle x_n \rangle = \langle y_n \rangle = 0$
  $\langle x_n^2 \rangle = \langle y_n^2 \rangle = n \langle \delta_{x,y}^2 \rangle$
  $\text{RMS}(x_n) = \text{RMS}(y_n) = \sqrt{n} \sqrt{\langle \delta_{x,y}^2 \rangle}$

Distance from the Starting Point



- r_i is the distance from the starting point to the position after step i.
- What are $\langle r_n \rangle$, $\langle r_n^2 \rangle$ and $\mathrm{RMS}(r_n)$?

The Expected Value for r^2

- For a single random walk:

$$r_n^2 = x_n^2 + y_n^2$$

- For two independent random variables, A and B

$$E(A + B) = E(A) + E(B)$$

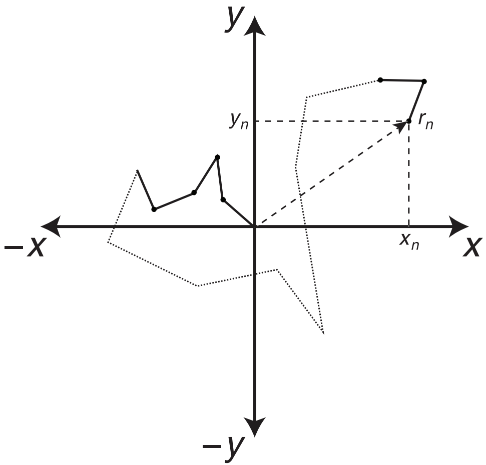
- The expected value of r_n^2:

$$E(r_n^2) = E(x_n^2) + E(y_n^2)$$

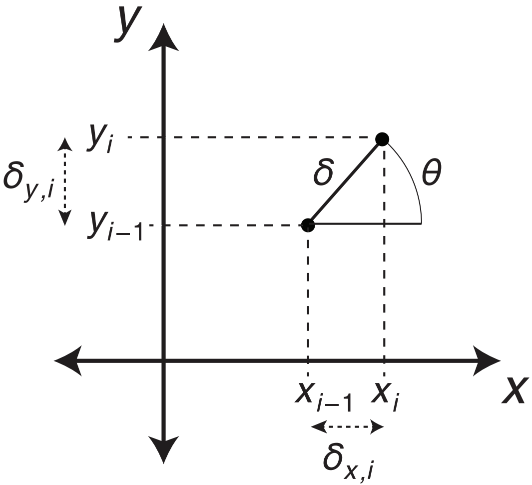
$$\langle r_n^2 \rangle = \langle x_n^2 \rangle + \langle y_n^2 \rangle$$

$$\langle r_n^2 \rangle = n\langle \delta_x^2 \rangle + n\langle \delta_y^2 \rangle$$

$$\langle r_n^2 \rangle = n2\langle \delta_{x,y}^2 \rangle$$



Calculating $\langle \delta_{x,y}^2 \rangle = \langle \delta_x^2 \rangle = \langle \delta_y^2 \rangle$



- Previously argued that

  $\langle \delta_{x,y}^2 \rangle < \delta^2$

- For a single step:

  $\delta_{x,i} = \delta \cos \theta$

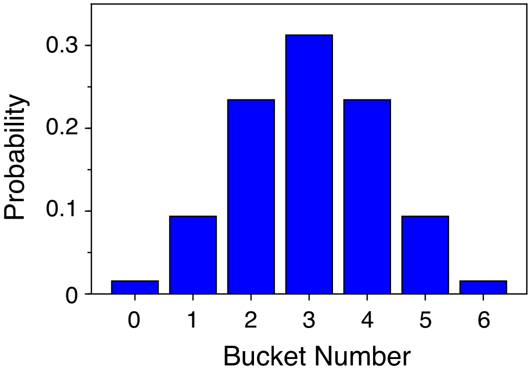
  $\delta_{y,i} = \delta \sin \theta$

- The key is the distributions of δ_x and δ_y values, which depend on the distribution of θ

Discrete Probability Distribution Functions

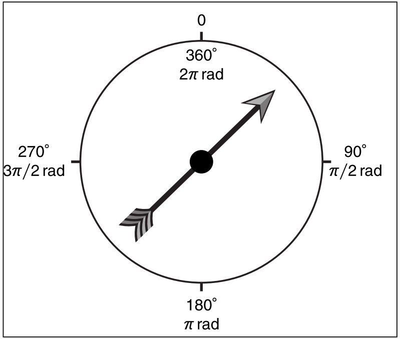
- So far, we have dealt with events with a finite number of discrete outcomes and random variables with discrete values.
- The probability distribution functions can be viewed as tables or bar graphs

Bucket No.	Probability
0	1/64
1	6/64
2	15/64
3	20/64
4	15/64
5	6/64
6	1/64



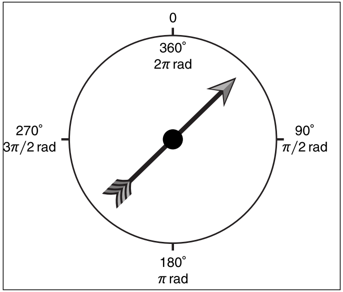
Introducing Continuous Probability Distribution Functions

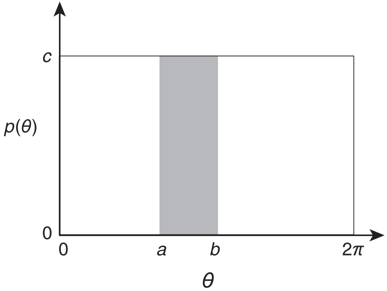
- A spinner to choose directions for the 2-dimensional random walk



- We could divide up the circle into a finite number of sectors.
  - Two sectors: Like flipping a coin
  - Six sectors: Like throwing a die
  - Lots of other possibilities
- OR, we can treat the result as a continuous variable from 0 to 2π rad

A Continuous Probability Distribution Function for the Spinner



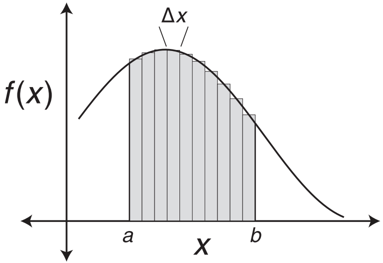


- θ is a continuous variable, with values from 0 to 2π.
- $p(\theta)$ is a function of θ, with a constant value, c, for all values of θ.
- Interpretation of $p(\theta)$: The integral

$$\int_a^b p(\theta)d\theta$$

is the probability that the spinner lands between the values a and b.

A Quick Refresher of Integrals (as "area under the curve")



- To approximate the area between the x-axis and the function $f(x)$, between $x = a$ and $x = b$:
  - Divide up the range $a \leq x \leq b$ into n segments $\Delta x = (b - a)/n$ wide.
  - Draw n rectangles Δx wide and $f(x_i)$ high.
  - Sum the areas of the rectangles

$$\text{area} = \sum_{i=1}^{n} f(x_i)\Delta x$$

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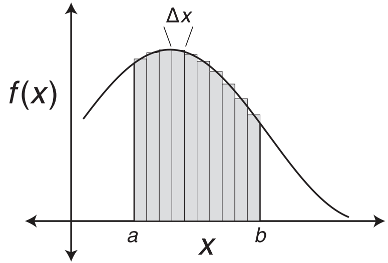

- Improve approximation by making Δx smaller (and n larger).
- If the function is "well behaved", Δx can be made infinitesimally small.
- The definite integral, from a to b with respect to x, is defined as:

$$\int_a^b f(x)dx = \lim_{\Delta x \to 0} \sum_{i=1}^{n} f(x_i)\Delta x$$

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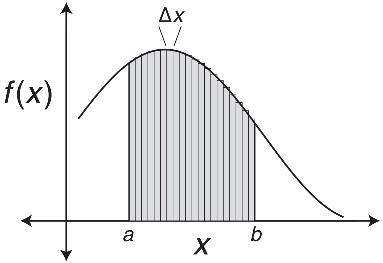

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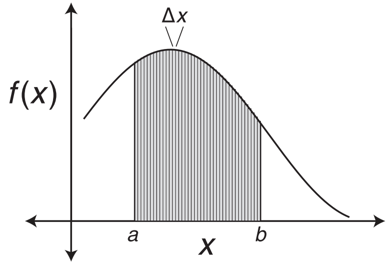

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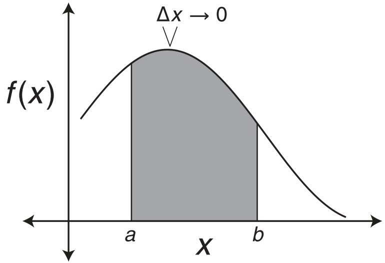

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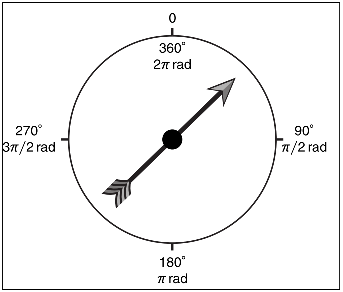
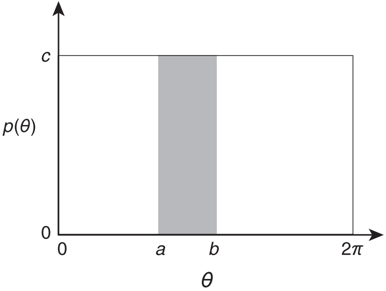
Back to the Continuous Probability Distribution Function (PDF) for the Spinner

- $p(\theta)$ is a function of θ, with a constant value, c, for all values of θ.



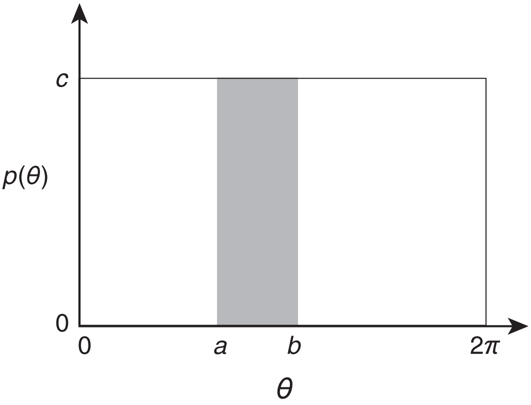


- The integral

$$\int_a^b p(\theta)d\theta$$

is the probability that the spinner lands between the values a and b.

An Important Constraint on a Continuous PDF



- To be properly "normalized", the integral

$$\int_0^{2\pi} p(\theta)d\theta$$

  must equal 1.

- Equivalent to the requirement for a discrete PDF that the sum of all probabilities be equal to 1.

- For the spinner pdf, setting the constant, c, equal to $1/(2\pi)$ normalizes the pdf.

$$p(\theta) = \frac{1}{2\pi}$$

The Expected Value for a Continuous PDF

- For a discrete random variable, x, with discrete PDF, $p(x)$, the expected value is:

$$E(x) = \sum_{i=1}^{n} xp(x)$$

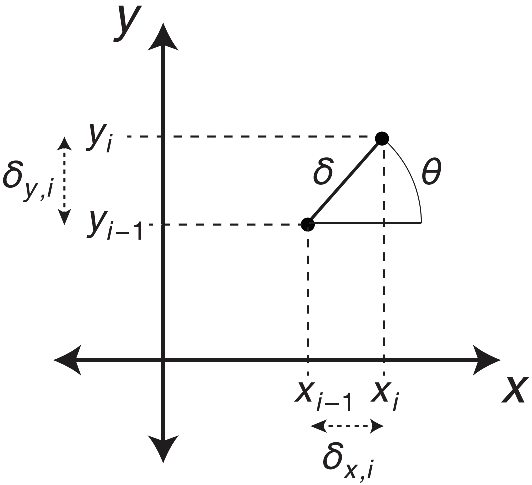
- For a continuous random variable, x, with range $x_1 \leq x \leq x_2$ and continuous PDF, $p(x)$, the expected value is:

$$E(x) = \int_{x_1}^{x_2} xp(x)dx$$

- For the spinner variable, θ:

$$E(\theta) = \int_{0}^{2\pi} \theta p(\theta)d\theta = \pi$$

The Expected Value of δ_x^2: $E(\delta_x^2) = \langle \delta_x^2 \rangle = \langle \delta_y^2 \rangle = \langle \delta_{x,y}^2 \rangle$



- $\delta_{x,i}^2$ as a function of θ_i

$$\delta_{x,i}^2 = (\delta \cos \theta)^2$$

- The expected value of δ_x^2

$$E(\delta_x^2) = \int_0^{2\pi} (\delta \cos(\theta))^2 p(\theta) d\theta$$

$$= \frac{\delta^2}{2}$$

- Now we can calculate the averages for the 2-dimensional random walk, in terms of n and δ.

Major Results for a Two-Dimensional Random Walk

For n steps of length δ:

- Displacement along the x- and y-axes (or any other direction):
  - Mean displacement: $\langle x_n \rangle = \langle y_n \rangle = 0$.
  - Mean-square displacement: $\langle x_n^2 \rangle = \langle y_n^2 \rangle = n\langle \delta_{x,y}^2 \rangle = n\delta^2/2$
  - RMS displacement: $\mathrm{RMS}(x_n) = \mathrm{RMS}(y_n) = \sqrt{n/2}\delta$
- Distance from starting point, r:
  - Mean-square displacement: $\langle r_n^2 \rangle == 2n\langle \delta_{x,y}^2 \rangle = n\delta^2$
  - RMS displacement: $\mathrm{RMS}(r_n) = \sqrt{n}\delta$
  - Just like the one-dimensional random walk!
  - Mean displacement: ?