

Physical Principles in Biology
Biology 3550
Fall 2017

Lecture 5:

Introduction to Probability

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Probability: Some Definitions

- Outcomes – Possible results of a probabilistic event
 - For a coin toss: coin lands heads-up (H) or tails-up (T)
 - For a roll of a six-sided die: The number of spots on the side that lands up (1, 2, 3, 4, 5 and 6)
 - Distinguished from “events”, to be defined shortly
- Probability – A number with a possible value from 0 to 1, associated with a single outcome.
 - $p = 0$: Outcome will never occur.
 - $p = 1$: Outcome will always occur.
 - The sum of the probabilities of all possible outcomes of an experiment must equal 1.
 - What do we mean by this? What is implied?

The Frequentist Interpretations of Probabilities

- If the same experiment is repeated a large number, N , times, an outcome with probability p will occur approximately $N \cdot p$ times.
- “Law of large numbers”
- Value of probabilities are defined by properties of the experiment.

The Sample Space, S

- Set of all possible outcomes

- For a single coin toss:

$$S = \{T, H\}$$

Curly braces are used to indicate sets.

- For two independent coin tosses:

$$S = \{(H, H), (H, T), (T, H), (T, T)\}$$

Ordered pairs representing the results of the two tosses.

- Are there other possible sample sets that could be defined for two coin tosses?

- The sample set must be complete, *i.e.*, it must include all possible outcomes.

- The elements in the sample set must not overlap.

- The sum of the probabilities for all of the elements in the sample set must equal 1.

Clicker Question #1

How many outcomes are there in the (simplest) sample set for three coin tosses?

1 3

2 4

3 6

4 8

5 10

The Sample Set for Three Coin Tosses

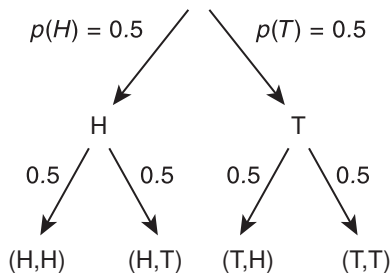
$$S = (H, H, H), (H, T, H), (H, H, T), (H, T, T) \\ (T, H, H), (T, T, H), (T, H, T), (T, T, T)\}$$

Events

- An event is a subset of the sample space.
- Some possible events defined for two coin tosses:
 - Two heads: $2H = \{(H, H)\}$
 - Two tails: $2T = \{(T, T)\}$
 - One heads and one tails: $1H1T = \{(H, T), (T, H)\}$
- The outcomes defined in the sample space are events, but additional events can usually be defined.
- Some other events that can be defined for two coin tosses:
 - One or more heads: $1^+H = \{(H, H), (H, T), (T, H)\}$
 - One or more tails: $1^+T = \{(H, T), (T, H), (T, T)\}$

Calculating Probabilities: Sequential Events

- Two coin tosses:



- Probabilities are multiplied

$$p((H, H)) = p(H)p(H) = 0.5 \times 0.5 = 0.25$$

$$p((H, T)) = p(H)p(T) = 0.5 \times 0.5 = 0.25$$

$$p((T, H)) = p(T)p(H) = 0.5 \times 0.5 = 0.25$$

$$p((T, T)) = p(T)p(T) = 0.5 \times 0.5 = 0.25$$

- Multiplication of probabilities is usually associated with “and”.

Clicker Question #2

What is the probability of three heads in three coin tosses?

1 0

2 $\frac{1}{8}$

3 $\frac{1}{3}$

4 $\frac{3}{8}$

5 $\frac{3}{4}$

6 1

The Probability of Three Heads in Three Coin Tosses

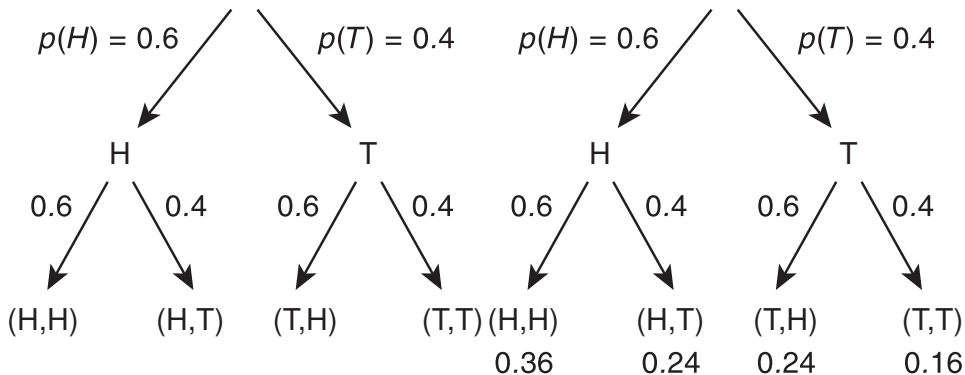
- Three sequential events

$$\xrightarrow{p=0.5} \text{H} \xrightarrow{p=0.5} \text{H} \xrightarrow{p=0.5} \text{H}$$

$$p = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$$

- Consider the sample set:
 - There are 8 elements in the sample set.
 - All outcomes in the sample set have equal probabilities (if the coin is fair).
 - For each outcome, $p = 1/8$.
 - Only one outcome has 3 heads, (H,H,H).

Two Tosses of a Bad Coin



- Sum of probabilities:

$$p((H, H)) + p((H, T)) + p((T, H)) + p((T, T)) = 1$$

Calculating Probabilities: Groups of Non-overlapping Events

- An event defined earlier for two coin tosses:

One heads and tails: $1H1T = \{(H, T), (T, H)\}$

- Probability is calculated as a sum:

$$\begin{aligned} p(1H1T) &= p((H, T)) + p((T, H)) \\ &= p(H)p(T) + p(T)p(H) \\ &= 0.5 \times 0.5 + 0.5 \times 0.5 \\ &= 0.25 + 0.25 = 0.5 \quad (\text{for a fair coin}) \end{aligned}$$

- This only works if the grouped events or outcomes do not overlap!
- Addition of probabilities is usually associated with “or”.

Another Example

- An event defined for two coin tosses:

One or more heads: $1^+H = \{(H, H), (H, T), (T, H)\}$

- Calculation of probability:

$$\begin{aligned} p(1^+H) &= p((H, H)) + p((H, T)) + p((T, H)) \\ &= 0.25 + 0.25 + 0.25 = 0.75 \quad (\text{for a fair coin}) \end{aligned}$$

- Another way:

- 1^+H includes all of the sample set, except (T, T)
- For the entire sample set, the sum of probabilities is 1.

$$\begin{aligned} p(1^+H) &= 1 - p((T, T)) \\ &= 1 - 0.25 = 0.75 \quad (\text{for a fair coin}) \end{aligned}$$

Clicker Question #3

What is the probability of exactly two heads in three coin tosses?

1 0

2 $1/8$

3 $1/4$

4 $3/8$

5 $1/2$

The Probability of Exactly Two Heads in Three Tosses

- The sample set for 3 coin tosses:

$$S = (H, H, H), (H, T, H), (H, H, T), (H, T, T) \\ (T, H, H), (T, T, H), (T, H, T), (T, T, T)\}$$

How many outcomes have exactly 2 heads:

$$(H, T, H), (H, H, T), (T, H, H)$$

- Another way:
 - An outcome with exactly 2 heads has exactly 1 tails!
 - There are three places in the sequence to place the 1 tails.
 - Three outcomes have exactly 2 heads.

Clicker Question #4

A coin has been tossed 10 times and has landed heads-up each time. What is the probability that it will land heads-up the next time?

- 1 0
- 2 Greater than zero but less than $1/2$
- 3 1
- 4 Greater than $1/2$ but less than 1
- 5 1

Clicker Question #5

The king has one sibling. What is the probability that the king's sibling is a sister?

1 $1/4$

2 $1/3$

3 $1/2$

4 $2/3$

5 $3/4$

Credit given for $1/2$, too

The King and His Sibling

- The sample set for two siblings:

$$S = \{(B, B), (B, S), (S, B), (S, S)\}$$

- But, we know that one of the siblings is a brother (the king).
- With this information, the sample set is reduced to:

$$S = \{(B, B), (B, S), (S, B)\}$$

- 2 of the three outcomes include a sister!
- Is there a hidden assumption here?