Physical Principles in Biology
Biology 3550
Fall 2017

Lecture 8

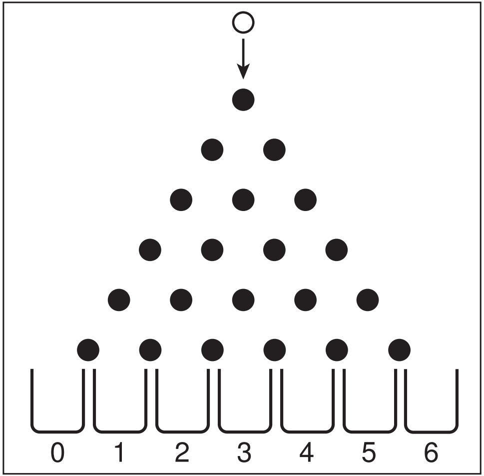
Plinko Probabilities, Part III:

Binomial Coefficients and the Bionomial Distribution Function

Friday, 8 September

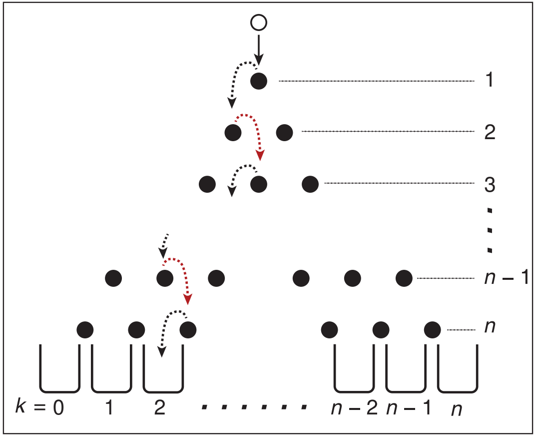
©David P. Goldenberg
University of Utah
goldenberg@biology.utah.edu

Probabilities for the Six-row Plinko



Bucket No.	Paths	Probability
0	1	$1/64 \approx 0.016$
1	6	$6/64 \approx 0.094$
2	15	$15/64 \approx 0.234$
3	20	$20/64 \approx 0.312$
4	15	$15/64 \approx 0.234$
5	6	$6/64 \approx 0.094$
6	1	$1/64 \approx 0.016$

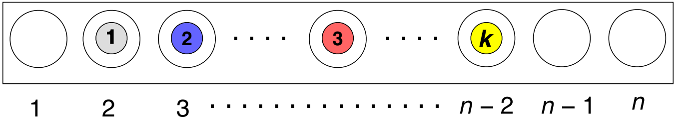
An n-row Plinko



- k = bucket number.
- To reach bucket k, ball must make k turns to the right and $n - k$ turns to the left.

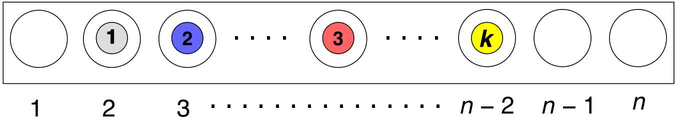
Beans and Cups

- For an n-row plinko, the number of paths to bucket k is the number of ways to place k labeled beans in n cups **in a single order**.



- To calculate this number:
  1. Calculate the number of ways to place k labeled beans in n cups, **in any order**.
  2. Calculate the number of ways to place k labeled beans in k cups, in any order.
  3. Divide result of 1 by result of 2.

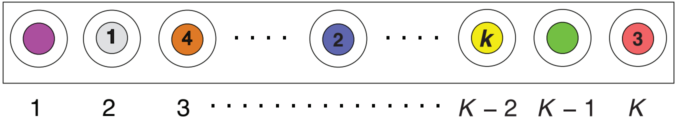
The Number of Ways to Place k Labeled Beans in n Cups, in Any Order



- The first bean can go in any of n cups.
- The second bean can go in any of the $n - 1$ cups that are left.
- The third bean can go in any of the $n - 2$ cups that are left.
- When it is time to find a place for the k^{th} bean:
  - $k - 1$ of the cups have beans in them.
  - The number of cups without beans is $n - (k - 1) = n - k + 1$
- The total number of ways to place the k labeled beans is:

$$n(n-1)(n-2)\cdots(n-k+1)$$

The Number of Ways to Place k Labeled Beans in k Cups, in Any Order



- The first bean can go in any of k cups.
- The second bean can go in any of the $k - 1$ cups that are left.
- The third bean can go in any of the $k - 2$ cups that are left.
- The k^{th} bean only has one place to go!
- The total number of ways to place the k labeled beans is:

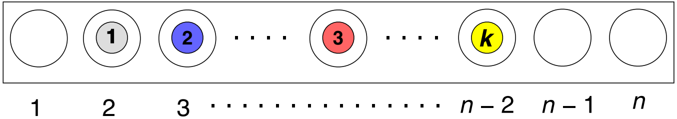
$$k(k-1)(k-2)\cdots 2\cdot 1$$

The Factorial Function

$$k! = \begin{cases} n(n-1)(n-2)\cdots 2 \cdot 1, & \text{if } k > 0; \\ 1, & \text{if } k = 0 \end{cases}$$

- $k!$ is only defined for integers greater than or equal to zero.
- Why is $0! = 1$?
- $k!$ is the number of distinct ways of placing k distinguishable objects in k positions.

The Number of Ways to Place k Labeled Beans in n Cups, in Any Order



- Previously showed that the total number of ways to place the k labeled beans is:

$$n(n-1)(n-2)\cdots(n-k+1)$$

- Can rewrite this as:

$$\frac{n(n-1)\cdots(n-k+1)(n-k)(n-k-1)\cdots 2\cdot 1}{(n-k)(n-k-1)\cdots 2\cdot 1} = \frac{n!}{(n-k)!}$$

Back to the Plinko

- For an n-row plinko, the number of paths to bucket k is the number of ways to place k labeled beans in n cups **in a single order**.
- The number of ordered ways to put k labeled beans in n cups is:

$$n(n-1)(n-2)\cdots(n-k+1) = \frac{n!}{(n-k)!}$$

- The number of ordered ways to place k labeled beans in k cups is:

$$k!$$

- The number of ways to place k labeled beans in n cups **in a single order** is:

$$\frac{n!}{k!(n-k)!}$$

- The universal plinko formula!

Test the Recipe on the Six-Row Plinko

- Paths to bucket 0: $\frac{n!}{k!(n-k)!} = \frac{n!}{0!n!} = \frac{1}{1} = 1$
- Paths to bucket 1: $\frac{n!}{k!(n-k)!} = \frac{n!}{1!(n-1)!} = \frac{n}{1} = n = 6$
- Paths to bucket 2: $\frac{n!}{k!(n-k)!} = \frac{n!}{2!(n-2)!} = \frac{n(n-1)}{2} = \frac{6 \cdot 5}{2} = 15$
- Paths to bucket 3: $\frac{n!}{k!(n-k)!} = \frac{6!}{3!3!} = \frac{720}{6 \cdot 6} = 20$

Test the Recipe on the Six-Row Plinko (contd.)

- Paths to bucket 4: $\dfrac{n!}{k!(n-k)!} = \dfrac{6!}{4!2!} = \dfrac{720}{24 \cdot 2} = 15$
- Paths to bucket 5: $\dfrac{n!}{k!(n-k)!} = \dfrac{6!}{5!1!} = \dfrac{720}{120 \cdot 1} = 6$
- Paths to bucket 6: $\dfrac{n!}{k!(n-k)!} = \dfrac{6!}{6!0!} = \dfrac{720}{720 \cdot 1} = 1$
- It works!
  Now we can do any size plinko!

n choose k

- The expression we just derived applies to much more than plinkos!
- The expression is often written as:

$$\binom{n}{k} = \frac{n!}{(n-k)!k!}$$

and spoken as "n choose k"

- From n objects, choose k of them (each only once) and either
  - Only a single order is valid (*e.g.,* turns in the plinko)

    Or
  - The order doesn't matter (*e.g.,* unlabeled beans).

Clicker Question #1

For a 5-row plinko, with 6 buckets labeled 0 to 5, how many paths are there to bucket 3?

1. 2
2. 4
3. 6
4. 8
5. 10

$$\binom{5}{3} = \frac{5!}{3!(5-3)!} = \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{(3 \cdot 2 \cdot 1)(2 \cdot 1)} = \frac{120}{12} = 10$$

Binomial Coefficients

- The series of numbers generated by

$$\binom{n}{k} = \frac{n!}{(n-k)!k!}$$

  for a specific value of n and increasing values of $k \leq n$ are called "binomial coefficients."

- The binomial coefficients arise in algebra in the expansion of a sum of two terms:

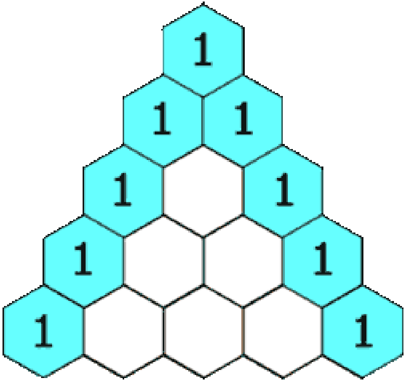
$$(a+b)^2 = a^2 + 2ab + b^2$$
$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$
$$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$
$$(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$$
$$(a+b)^6 = a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$$

Pascal's Triangle

	Row
1	0
1 1	1
1 2 1	2
1 3 3 1	3
1 4 6 4 1	4
1 5 10 10 5 1	5
1 6 15 20 15 6 1	6 ,

- Blaise Pascal, French mathematician, 1623–1662
- Triangle was known long before Pascal's time, but Pascal wrote a book about it.

Pascal's Triangle



- Start with 1s on left and right sides.
- Calculate other elements by adding two values above.

Animation from `https://en.wikipedia.org/wiki/Pascal%27s_triangle`

Probabilities for an n-row Plinko

- The total number of paths is 2^n.
- If each turn to the right or left is equally probable, the probabilities of all paths are equal, and the probability of each path is:

$$p = \frac{1}{2^n} = 2^{-n}$$

- The probability of a ball landing in bucket k is the number of paths to the bucket multiplied by the probability of each path:

$$p(k) = \frac{n!}{k!(n-k)!} \cdot 2^{-n}$$

Clicker Question #2

For a 7-row plinko, with 8 buckets labeled 0 to 7, what is the probability of a ball landing in bucket 1?

(There's a hard way and an easy way!)

1. ~ 0.01
2. ~ 0.05
3. ~ 0.1
4. ~ 0.15
5. ~ 0.2

$$p(1) = \frac{7!}{1!(n-1)!} \cdot 2^{-7} = \frac{7!}{6!} \cdot 2^{-7} = 7 \cdot 2^{-7}$$

What if the Plinko is Biased?

- Suppose that each peg in the plinko has been “fixed”, so that the probability of a left turn is 0.4 and the probability of a right turn is 0.6.
- For each of the paths to bucket k, there are k right turns and $(n - k)$ left turns.
- For each individual path to bucket k, the probability is:

$$0.6^k \times 0.4^{(n-k)}$$

- The total probability of a ball falling in bucket k is:

$$p(k) = \frac{n!}{k!(n-k)!} \times 0.6^k \times 0.4^{(n-k)}$$

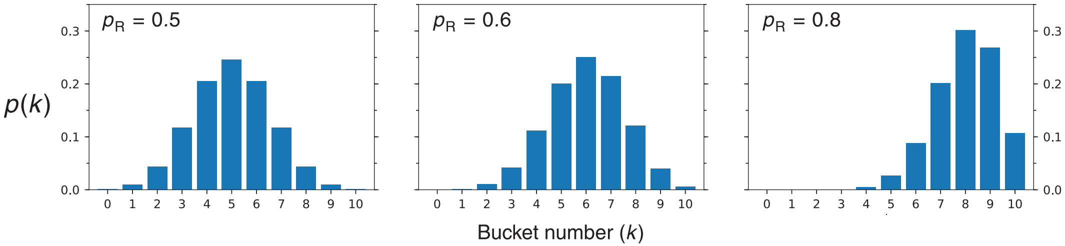
The Binomial Probability Distribution Function

- The general formulation:
  $p(k; n, p)$ is the probability of k successes in n successive binary (yes/no) trials when the probability of success in each trial is p.
- The probability function:

$$p(k; n, p) = \frac{n!}{k!(n-k)!} p^k (1-p)^{(n-k)}$$

- Some applications beyond plinkos:
  - Number of heads in n successive coin tosses.
  - Number of successes in prescribing a medication to a series of patients with the same condition.
  - Probability of surviving n potentially deadly events.
    $p(n; n, p)$, were p is the probability of surviving each event

Probabilities for a Biased 10-row Plinko



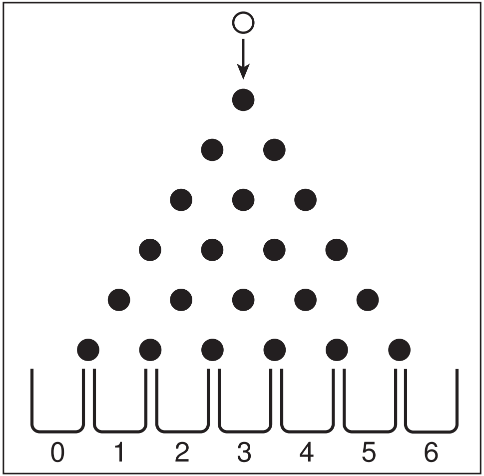
- Biased pegs "push" balls to the right.
- Probability (number of paths) "draws" balls to the center.
- Can you think of physical processes like this?

Playing Plinko for Cash

- Suppose that I let you put a ball in the 6-row plinko, and I agree to pay you k dollars if the ball lands in bucket k.
- This is probably going to cost me money!
- How much should I charge you to play?
- How much, on average, am I going to have to pay?

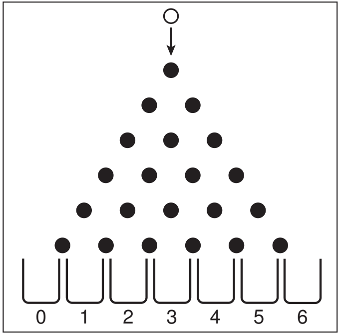
Clicker Question #3

How much should I charge you to play my plinko game (to break even)?
All answers count for now.



1. $1
2. $2
3. $3
4. $4
5. $6
6. $7

An Intuitive Solution



- Buckets 0 and 6 have equal probabilities. The average payout for these two is $3.
- Buckets 1 and 5 have equal probabilities. The average payout for these two is $3.
- Buckets 2 and 4 have equal probabilities. The average payout for these two is $3.
- The payout for bucket 3 is $3.
- The overall average payout is $3.

Random Variables

- Definition: A variable that is assigned a value for each possible outcome or event for a probabilistic process.
- Examples:
  - For a coin toss, we could assign a random variable, x, the value of 1 for heads or 0 for tails.
  - For n successive coin tosses, we could define x to be the number of heads.
  - For the Plinko, we can define the random variable, x, as the number of the bucket that the ball lands in.
    But, we could define other random variables, too.

The Expected Value or Expectation

For a random process that has n possible outcomes (or a complete set of n non-overlapping events):

- The random variable, x, has values of x_k for $k = 1, 2, 3 \ldots n$
- The n possible outcomes (or events) have probabilities of $p(k)$, for $k = 1, 2, 3 \ldots n$
- The expected value is defined as:

$$E = \sum_{k=1}^{n} p(k) x_k$$

- If the process is repeated a large number of times, the average value of x will approach E.
- For a game of chance, if x_k is the number of dollars paid out for outcome (or event), k, E is the average payout.