Physical Principles in Biology
Biology 3550
Fall 2017

Lecture 9

Random Variables and Expected Values:
and
Introduction to Random Walks

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Playing Plinko for Cash

- Suppose that I let you put a ball in the 6-row plinko, and I agree to pay you k dollars if the ball lands in bucket k.
- This is probably going to cost me money!
- How much should I charge you to play?
- How much, on average, am I going to have to pay?

Random Variables

- Definition: A variable that is assigned a value for each possible outcome or event for a probabilistic process.
- Examples:
  - For a coin toss, we could assign a random variable, x, the value of 1 for heads or 0 for tails.
  - For n successive coin tosses, we could define x to be the number of heads.
  - For the Plinko, we can define the random variable, x, as the number of the bucket that the ball lands in.
    But, we could define other random variables, too.

The Expected Value or Expectation

For a random process that has n possible outcomes (or a complete set of n non-overlapping events):

- The random variable, x, has values of x_k for $k = 1, 2, 3 \ldots n$
- The n possible outcomes (or events) have probabilities of $p(k)$, for $k = 1, 2, 3 \ldots n$
- The expected value is defined as:

$$E = \sum_{k=1}^{n} p(k) x_k$$

- If the process is repeated a large number of times, the average value of x will approach E.
- For a game of chance, if x_k is the number of dollars paid out for outcome (or event), k, E is the average payout.

Expected Value for the Unbiased Six-row Plinko

Bucket	x	$p(x)$	$p(x)x$
0	0	1/64	0
1	1	6/64	6/64
2	2	15/64	30/64
3	3	20/64	60/64
4	4	15/64	60/64
5	5	6/64	30/64
6	6	1/64	6/64
Total		1	192/64 =3

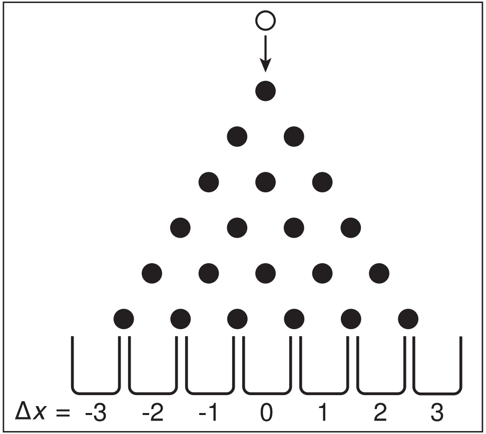
Expected Value for a Biased Six-row Plinko: $p(\text{right}) = 0.6$

Bucket	x	$p(x)$	$p(x)x$
0	0	0.004	0
1	1	0.037	0.037
2	2	0.138	0.276
3	3	0.276	0.829
4	4	0.311	1.244
5	5	0.187	0.933
6	6	0.046	0.280
Total		1	3.6

Expected Value for a Biased Six-row Plinko: $p(\text{right}) = 0.4$

Bucket	x	$p(x)$	$p(x)x$
0	0	0.046	0
1	1	0.187	0.187
2	2	0.311	0.622
3	3	0.276	0.829
4	4	0.138	0.553
5	5	0.037	0.184
6	6	0.004	0.0246
Total		1	2.4

Another Random Variable for the Plinko, Δx



- Δx represents the position of the bucket, relative to the central bucket.

Expected Value of Δx for the Unbiased Six-row Plinko

Bucket	Δx	$p(\Delta x)$	$p(\Delta x)\Delta x$
0	-3	1/64	-3/64
1	-2	6/64	-12/64
2	-1	15/64	-15/64
3	0	20/64	0
4	1	15/64	15/64
5	2	6/64	12/64
6	3	1/64	3/64
Total		1	0

Expected Value of Δx for a Biased Six-row Plinko: $p(\text{right}) = 0.6$

Bucket	Δx	$p(\Delta x)$	$p(\Delta x)\Delta x$
0	-3	0.004	-0.012
1	-2	0.037	-0.074
2	-1	0.138	-0.138
3	0	0.276	0
4	1	0.311	0.311
5	2	0.186	0.373
6	3	0.046	0.139
Total		1	0.6

Notice:

- The two random variables are related according to:

$$\Delta x = x - 3$$

- For the unbiased 6-row plinko:

$$E(x) = 3$$
$$E(\Delta x) = 0$$

- For the biased 6-row plinko:

$$E(x) = 3.6$$
$$E(\Delta x) = 0.6$$

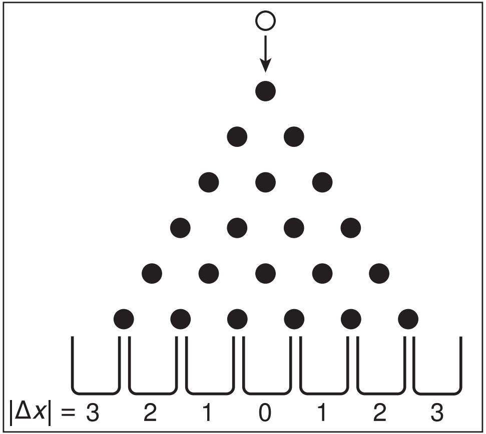
- In general, if x is a random variable, and a is a constant:

$$E(a + x) = a + E(x)$$

- Also:

$$E(ax) = aE(x)$$

Another Random Variable for the Plinko, $|\Delta x|$



- $|\Delta x|$ represents the average distance from the central bucket.

Expected Value of $|\Delta x|$ for the Unbiased Six-row Plinko

Bucket	$\|\Delta x\|$	$p(\|\Delta x\|)$	$p(\|\Delta x\|)\,\|\Delta x\|$
0	3	1/64	3/64
1	2	6/64	12/64
2	1	15/64	15/64
3	0	20/64	0
4	1	15/64	15/64
5	2	6/64	12/64
6	3	1/64	3/64
Total		1	$60/64 \approx 0.94$

Expected Value of $|\Delta x|$ for a Biased Six-row Plinko: $p(\text{right}) = 0.6$

Bucket	$\|\Delta x\|$	$p(\|\Delta x\|)$	$p(\|\Delta x\|)\,\|\Delta x\|$
0	3	0.004	0.012
1	2	0.037	0.074
2	1	0.138	0.138
3	0	0.276	0
4	1	0.311	0.311
5	2	0.186	0.373
6	3	0.046	0.140
Total		1	1.05

Question #1

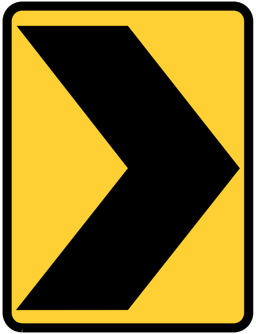
What is the expected value of $|\Delta x|$ for a biased six-row plinko with $p(\text{right}) = 0.4$?

1. -1.05
2. -0.94
3. 0
4. 0.94
5. 1.05

Expected Value of $|\Delta x|$ for a Biased Six-row Plinko: $p(\text{right}) = 0.4$

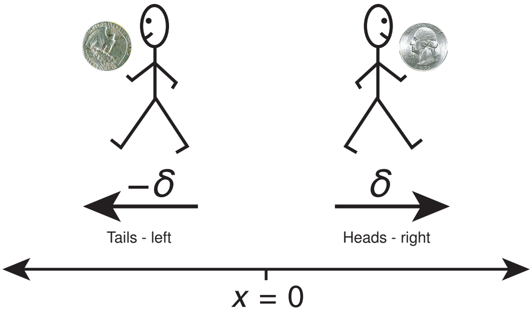
Bucket	$\|\Delta x\|$	$p(\|\Delta x\|)$	$p(\|\Delta x\|)\,\|\Delta x\|$
0	3	0.046	0.140
1	2	0.187	0.373
2	1	0.311	0.311
3	0	0.276	0
4	1	0.138	0.138
5	2	0.037	0.074
6	3	0.004	0.012
Total		1	1.05

Warning!



Direction Change

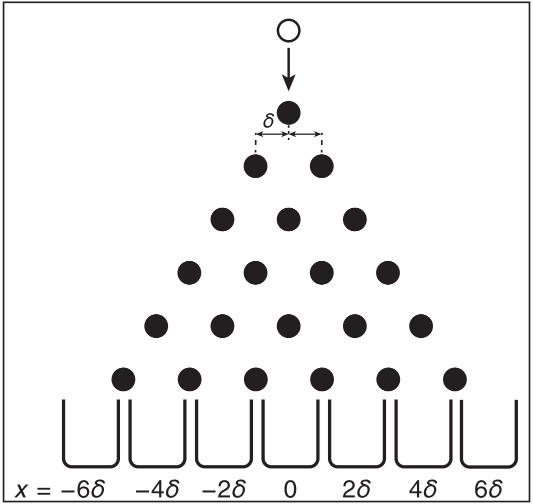
A Random Walk in One Dimension



1. Start at position $x = 0$.
2. Flip coin.
   - Heads, take step of length δ to the right.
   - Tails, take step of length δ to the left.
3. Repeat 2 another $(n - 1)$ times.

- Final position is $x(n)$.
- Generally expect a distribution of $x(n)$ if the random walk is repeated a large number, N, of times.

Like a Plinko, with variable x



- x represents the position of the bucket, relative to the central bucket.

What Do We Know About $x(n)$, the End Point?

- The maximum value of $x(n)$ is δn.
- The minimum value of $x(n)$ is $-\delta n$.
- If we repeat the random walk many times, the distribution of $x(n)$ will be binomial.
- But, if n is very large, calculating the binomial distribution will be difficult!

Calculate The Average Final Position (The Expected Value of $x(n)$)

- For a single random walk, the final position will be:

$$x(n) = \sum_{i=1}^{n} \delta_i$$

  where i is the step number, and δ_i is either $-\delta$ or $+\delta$, with probabilities $p(+\delta)$ and $p(-\delta)$.

- For each step, δ_i is a random variable, with an expected value, $E(\delta_i)$:

$$\begin{aligned} E(\delta_i) &= \delta p(+\delta) - \delta p(-\delta) \\ &= \delta p(+\delta) - \delta(1 - p(+\delta)) \\ &= \delta p(+\delta) + p(+\delta) - \delta \\ &= 2\delta p(+\delta) - \delta = \delta\big(2p(+\delta) - 1\big) \end{aligned}$$

Clicker Question #2

If the random-walk step size is 0.5 m, and the probability of a forward step, $p(+\delta)$, is 0.3, what is the expected value for the displacement in a single step, $E(\delta_i)$?

1. -0.5 m
2. -0.2 m
3. 0 m
4. 0.2 m
5. 0.4 m

$$
\begin{aligned}
E(\delta_i) &= \delta p(+\delta) - \delta p(-\delta) \\
&= 0.5\,\mathrm{m} \cdot 0.3 - 0.5\,\mathrm{m} \cdot 0.7 \\
&= 0.5\,\mathrm{m}(0.3 - 0.7) = -0.2
\end{aligned}
$$

Calculating The Expected Value of $x(n)$

- An important theorem: If x and y are two independent random variables, then the expected value of the sum is calculated as:

$$E(x+y) = E(x) + E(y)$$

- Since:

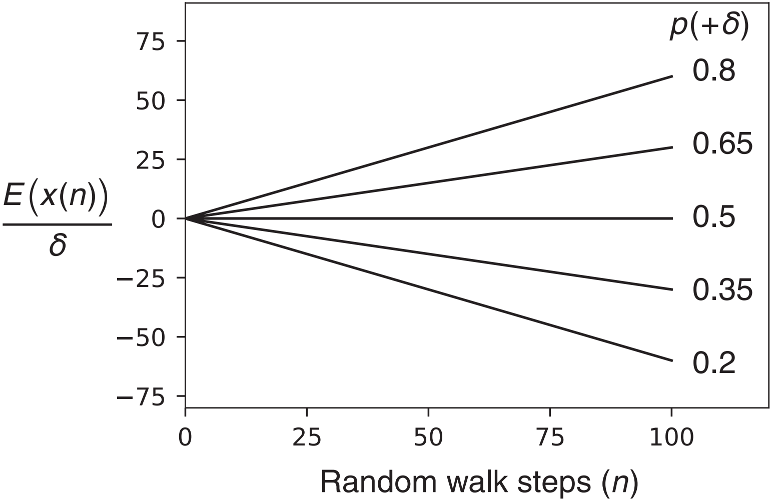
$$x(n) = \sum_{i=1}^{n} \delta_i$$

The expected value of $x(n)$ is calculated as:

$$E\big(x(n)\big) = \sum_{i=1}^{n} E(\delta_i)$$

$$= \sum_{i=1}^{n} \delta\big(2p(+\delta) - 1\big) = n\delta\big(2p(+\delta) - 1\big)$$

Expected Value of $x(n)$ for a One-dimensional Random Walk



Some Different Kinds of Average

For N random walks of n steps each:

- The mean:

$$\langle x(n)\rangle = \frac{1}{N}\sum_{j=1}^{N} x_i(n), \text{ approaches } E\big(x(n)\big)$$

- The mean-square:

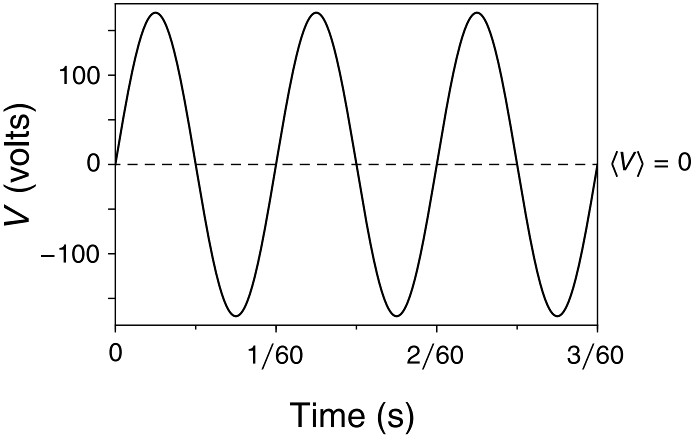
$$\langle x(n)^2\rangle = \frac{1}{N}\sum_{j=1}^{N} x_i(n)^2$$

- The root-mean-square (RMS):

$$\mathrm{RMS}(x(n)) = \sqrt{\langle x(n)^2\rangle} = \sqrt{\frac{1}{N}\sum_{j=1}^{N} x_i(n)^2}$$

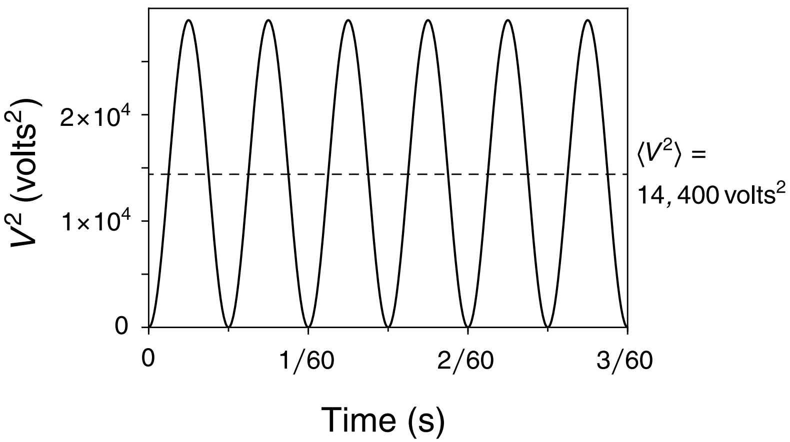
An Application of Mean-square and Root-mean-square Averages: Household Power (US)

Voltage versus time



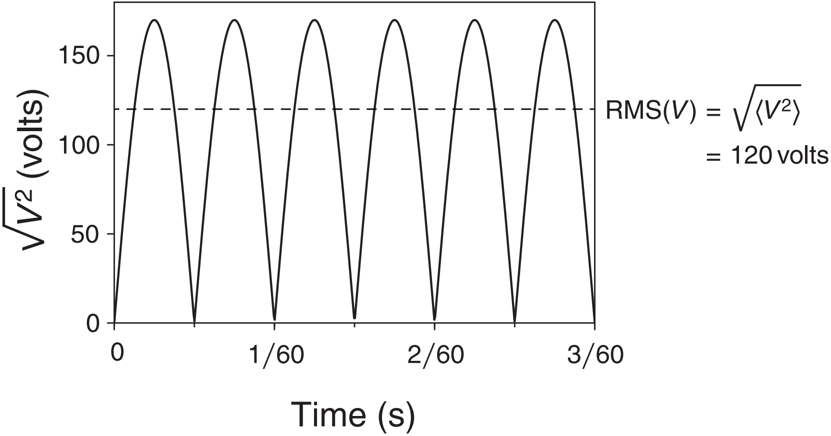
An Application of Mean-square and Root-mean-square Averages: Household Power (US)

Voltage squared versus time



An Application of Mean-square and Root-mean-square Averages: Household Power (US)

$\sqrt{V^2}$ versus time



Clicker Question #3

For the numbers: $-4, 2, -3, 1, 5,$
Calculate the root-mean-square average

1. ~ 0.2
2. ~ 1.5
3. ~ 2.9
4. ~ 3.3
5. ~ 4.8

$$\text{RMS} = \sqrt{\frac{-4^2 + 2^2 + -3^2 + 1^2 + 5^2}{5}} = \sqrt{\frac{16 + 4 + 9 + 1 + 25}{5}} = \sqrt{\frac{55}{5}} = \sqrt{11}$$

Calculating the Mean-Square Displacement for a 1-d Random Walk

- For a single random walk, the final position will be:

$$x(n) = \sum_{i=1}^{n} \delta_i$$

  where i is the step number, and δ_i is either $-\delta$ or $+\delta$, with equal probability (if the coin is fair), for each step.

- We can also express $x(n)$ in terms of the position after the next-to-last step, $x(n-1)$:

$$x(n) = x(n-1) + \delta_n$$

Calculating The Mean-Square Displacement

- If we do a large number, N, of random walks, the mean-square displacement, $\langle x \rangle$, will be:

$$\langle x(n)^2 \rangle = \frac{1}{N} \sum_{j=1}^{N} x_j(n)^2 = \frac{1}{N} \sum_{j=1}^{N} \left(\sum_{i=1}^{n} \delta_{j,i} \right)^2$$

  where j is the random walk number, and $\delta_{j,i}$ is the displacement for step i of walk j.

- We can also write the mean-square average as:

$$\langle x(n)^2 \rangle = \frac{1}{N} \sum_{j=1}^{N} \left(x_j(n-1) + \delta_{j,n} \right)^2$$

$$= \frac{1}{N} \sum_{j=1}^{N} \left(x_j(n-1)^2 + 2x_j(n-1)\delta_{j,n} + \delta_{j,n}^2 \right)$$

  $\delta_{n,j}$ is the change in position for the last step in random walk j.